

Original Paper

Nonlinear Consequences of Affine Failure Across Mathematics and Physics: Finite Refinement, Persistent Transport, and Exact Projection Theorems

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Abstract: Finite refinement, event transport, and observation are developed as distinct but compatible structures. Coherent histories carry a compact ultrametric completion; persistent sign classes define an intrinsic carrier; regular branching calibrates its Hausdorff dimension; and preservation of observed fibers characterizes exact quotient dynamics. Scalar completion can retain a positional readout while losing the variation or ancestry of its generating transport. Prime-word arithmetic realizes this distinction through exact successive-prime sieve and Mellin handoffs. The normalized Mellin state is stationary exactly at nontrivial zeta zeros and recovers the canonical exponent; an exact stationary-state defect characterizes localization in the canonical coordinate. In incompressible transport, a closed face-flux cochain, retained wake deposition, gauge-fixed pressure, and discrete Hodge viscosity determine a finite one-event law. Compatible Whitney reconstruction yields an exact divergence-free projected-affine balance. Successive wakes retain and re-expose earlier remnants. The anticonformal source supplies the open event, successor kernel, and history-conditioned clock. Its positive duration bound proves nonaccumulation and global native continuation through every finite clock interval at fixed resolution. Exact residual elimination, a general returning-aperture completion, and transported metric identities give complete-state estimates. An inherited-projection recurrence has the sharp uniform bound $(73\pi/192)\|x_0\|$, derived without an independent remainder budget. Compatible periodic and impermeable-wall complexes, nonzero closed-sector nonlinear flux, and finite-horizon observation closure

are constructed. A finite circular vector projection supplies a global smooth fluid realization with an exact commuting event map and retained pressure-absorbable stress. The computational realization distinguishes exact motion from existential closure and from required readout. Two translated elliptic observations recover a point. For the fixed doubling orbit on $y^2 = x^3 - 2$, cancellation-free reduced-coordinate growth gives an explicit binary-output length $L_m = \Theta(4^m)$, an unconditional $\Omega(L_m)$ time lower bound, and a constructive $O(L_m \log(L_m + 2))$ upper bound. Boolean gate lifts, finite closure ledgers, and algebraic certificate constructions separate model-specific nonabsorption from actual generation cost. Normalized cost ledgers give exact transfer theorems for separately specified decision interfaces. The arithmetic and geometric extensions specify the maps needed for each application. A retained-deficit event constructs sector frequencies with a finite deterministic error bound, and planar similarity sectors supply quadratic weights.

Keywords: Persistent remainders; affine failure; anticonformal geometry; prime-sieve dynamics; discrete fluid transport

1. Introduction: finite motion and affine readout

An observed line, a scalar value, or a reconstructed field records selected features of a mathematical object. It need not record how that object was generated, which finite history it carries, or what information its next admissible event requires. Distinct histories may share a present image, while a faithful positional description may leave accumulated transport information unresolved. The question developed here is therefore not only what an observation represents, but what it preserves when the underlying object continues to evolve.

The distinction concerns the generating registry and its coordinate image. An affine or scalar readout may be exact for the quantities it is designed to report without exhausting the structure whose events it records. A finite staircase and its diagonal share endpoints while having different path lengths; an observed wake state can omit ancestry while still obeying a closed successor law. These are different forms of information loss, and their separation is central to the argument. Failure to reconstruct a history is not the same as failure to predict its next observed state. The finite constructions below determine when each occurs.

In the Chronoscalar formulation developed here, finite ordered histories precede their

geometric and continuous representations. Motion means a succession of admissible events, not necessarily displacement along a pre-existing smooth curve. Refinement relates finite descriptions of those events and the distinctions they retain. Its infinite branch boundary is a mathematical completion used to compare coherent refinements; every occupied history entering a finite calculation remains finite. Coordinates, interpolations, and continuum fields are introduced through specified readout maps rather than substituted for the event structure at the outset.

The retained information passed from one event to another is called the *wake*. Depending on the realization, it may record an inherited load, a circulation ancestry, or a future-relevant distinction that the present observation does not resolve. The relevant question is whether an observation can carry the event law exactly. If two hidden states have the same present image but different successor images, that observation is insufficient for the update. If every observed fiber is transported into a successor fiber, the quotient evolution is well defined even when the hidden history is not recoverable. This distinction between exact observation and complete reconstruction supplies the common thread of the manuscript.

The finite second variation introduced below compares consecutive increments along an admissible three-state chain: equal increments cancel, while a nonzero value detects departure from that local affine pattern. Its sign can then be followed along coherent descendants. The refinement metric determines intrinsic separation and dimension; an explicit carrier realization supplies any additional geometric structure; and a separate compatibility theorem determines whether the observation supports the motion. These steps have distinct mathematical roles. In particular, a sign pattern does not by itself construct a geometric carrier, and nonlinearity alone does not determine a Hausdorff dimension or a computational cost.

Prime-word arithmetic provides one extended realization. Successive admission of primes changes a finite counting staircase through an exact remainder handoff. The Mellin state is then a readout of that arithmetic evolution. Every prime occurs in the ordered admission registry, and every finite remainder is retained by the exact handoff. Prime-threshold stationarity characterizes the nontrivial zeta zeros. The location of those zeros in the complex plane is a different statement; the coordinate discussion is kept distinct from the finite propagation law. This development occupies Part II.

Incompressible transport provides another realization. A finite cochain state, retained wake deposition, a pressure compatibility condition, and discrete viscosity define the event ledger before continuum reconstruction. Compatible reconstruction gives exact divergence freedom and a projected-affine balance. The successive-wake argument tracks every inherited remnant through its later handoffs. The anticonformal source equations identify the open response and its successor kernel; their clock law gives a positive step floor and hence global native continuation at every fixed resolution. The general passive-aperture completion

preserves the full local flux budget, including return channels. The Wallis–Gregory estimate applies to its specified recurrence, and explicit spatial constructions supply the impermeable-wall complex. Exact observed closure is checked on full fibers, including any memory needed by the next event. Finite circular carriers projected componentwise give a smooth vector field. The explicit circular family proves the classical fluid equation and exact event commutation, with nonzero retained stress absorbed into pressure. The full fluid construction in Part VI therefore makes the distinction between event dynamics and observed fields operational rather than merely descriptive.

The computational development asks a further question: what work is required to realize an exact observation? Two translated elliptic observations recover a point, but a fixed number of coordinates does not bound their growing binary length. For the fixed doubling orbit on $y^2 = x^3 - 2$, Theorem 39.4 proves an unconditional exponential lower bound for the stipulated explicit-coordinate output and gives a constructive upper bound within a logarithmic factor of its output length. The bound holds for every correct deterministic implementation with unrestricted internal representations. The later decision-closure section gives transfer criteria for deciding whether an accepting state exists from a succinct description.

The arithmetic, geometric, logical, and physical sections specify the event maps, observations, and conserved quantities of their respective constructions. Part I begins the formal development with finite refinement, coherent branches, and intrinsic support, then proves the projection and transport statements used throughout. The purpose of that foundation is to make precise how a faithful readout can remain incomplete, and how the information it omits may matter to subsequent motion or to the cost of an exact realization.

Part I

Finite refinement and projection

2. Admissible finite refinement and the branch boundary

Definition 2.1 (Finite refinement system). An admissible refinement system is a sequence

$$\mathcal{R} = (S_n, \mathcal{C}_n, R_n, T_n, h_n)_{n \geq 0}$$

where S_n is a finite state set, $\mathcal{C}_n$ a finite family of admissible three-point chains, $T_n : S_n \rightarrow \mathbb{R}$ an ordering readout,

$$R_n : S_n \longrightarrow 2^{S_{n+1}}, \quad R_n(p) \neq \emptyset,$$

and $h_n > 0$ is an observational resolution with $h_n \downarrow 0$ in the comparison family. Physical histories used in an experiment or finite calculation remain finite; the infinite object below is the mathematical boundary of the refinement family.

Definition 2.2 (Coherent branch and primary second variation). A coherent branch is a sequence $p = (p_0, p_1, \dots)$ with $p_{n+1} \in R_n(p_n)$. The set of all branches is denoted $\partial\mathcal{R}$. For a chain $c = (p_{i-1}, p_i, p_{i+1}) \in \mathcal{C}_n$ define

$$\Delta_{T_n}^2(c) = T_n(p_{i+1}) - 2T_n(p_i) + T_n(p_{i-1}).$$

Fix $0 < \rho < 1$. If $p \neq q$ are branches let

$$N(p, q) = \min\{n \geq 0 : p_n \neq q_n\}, \quad d_{\text{adm}}(p, q) = \rho^{N(p, q)},$$

and put $d_{\text{adm}}(p, p) = 0$.

Proposition 2.3 (Refinement ultrametric and compactness). d_{adm} is an ultrametric. If the refinement system is finitely branching and has no terminal descendant, then $(\partial\mathcal{R}, d_{\text{adm}})$ is compact.

Proof. If p and q agree through level $m - 1$ and q and r agree through that level, then p and r do also. Hence

$$N(p, r) \geq \min\{N(p, q), N(q, r)\},$$

which is equivalent to the ultrametric inequality. At level n there are finitely many prefixes, and their cylinder sets are balls of radius ρ^n , so the space is totally bounded. A Cauchy sequence stabilizes at each finite prefix and therefore defines a coherent branch; the space is complete. Complete plus totally bounded gives compactness. $\square$

3. Persistent sign branches and an actual existence theorem

Definition 3.1 (Persistent sign branches). For $\sigma \in \{-1, +1\}$ let $B_\sigma \subset \partial\mathcal{R}$ consist of branches p for which there are $N_\sigma < \infty$ and chains $c_n \in \mathcal{C}_n$ containing p_n such that

$$\operatorname{sgn} \Delta_{T_n}^2(c_n) = \sigma \quad (n \geq N_\sigma).$$

The intrinsic sign carrier is

$$L_\infty := \overline{B_+}^{d_{\text{adm}}} \cap \overline{B_-}^{d_{\text{adm}}}.$$

Persistence is a property of one coherent descendant history: after a finite initial segment, the chosen sign is witnessed at every subsequent level along that branch.

Theorem 3.2 (Nested-cylinder carrier existence). Suppose $(\partial\mathcal{R}, d_{\text{adm}})$ is compact. Let

$$U_0 \supset U_1 \supset U_2 \supset \cdots$$

be nonempty refinement cylinders with $\operatorname{diam}(U_n) \leq \rho^n$. Assume

$$U_n \cap B_+ \neq \emptyset, \quad U_n \cap B_- \neq \emptyset$$

for every n . Then $L_\infty \neq \emptyset$.

Proof. Choose $p_n^+ \in U_n \cap B_+$ and $p_n^- \in U_n \cap B_-$. The nested cylinders have diameters tending to zero, so by compactness their intersection consists of a single branch p . Let V be any d_{adm} -neighborhood of p . For sufficiently large n , $U_n \subset V$. Then V contains both p_n^+ and p_n^- . Thus every neighborhood of p meets both persistent sign classes, and

$$p \in \overline{B_+} \cap \overline{B_-} = L_\infty.$$

□

Remark 3.3. The nested-cylinder construction produces a carrier point from coherent coexistence at successive finite depths. Its input is the family of shrinking cylinders, and its output is a branch belonging to the closure of both persistent sign classes.

Theorem 3.4 (Explicit persistent-sign realization). For every integer $b \geq 2$ and $0 < \rho < 1$, there is a finite refinement system whose two persistent sign classes are both dense in its entire branch boundary. Its carrier is nonempty and has dimension $\log b / (-\log \rho)$.

Proof. Let $A = \{0, \dots, b-1\}$, $S_0 = \{(\emptyset, 0)\}$ and, for $n \geq 1$, $S_n = A^n \times \{-1, 0, 1\}$. The root refines to all $(a, 0)$; thereafter (w, t) refines to the b states (wa, t) . At level $n \geq 1$ use the chains $((w, -1), (w, 0), (w, 1))$ and put

$$T_n(w, t) = \frac{1}{2} \sigma(w_n) t^2, \quad \sigma(0) = 1, \quad \sigma(a) = -1 \ (a \neq 0), \quad h_n = \rho^n.$$

The second variation on this chain is exactly $\sigma(w_n)$. Every branch from the root has marker zero. After any finite prefix, choosing all later letters zero gives a positive persistent branch, and choosing all later letters one gives a negative persistent branch. Thus both classes meet every cylinder and $L_\infty = \partial\mathcal{R} \cong A^\mathbb{N}$.

A length- n cylinder has b^{-n} Bernoulli mass and diameter ρ^{n+1} . The common extra factor ρ in the metric does not change dimension. The cylinder-cover and mass-distribution arguments of the next section give exactly $\log b/(-\log \rho)$. All state layers and chain families are finite, and every state has a successor. This constructs the coexistence and regular-branching hypotheses for every chosen pair (b, ρ) in the stated range. $\square$

4. Intrinsic Hausdorff support

Definition 4.1. The intrinsic Hausdorff dimension of the carrier is

$$D_H := \dim_H^{d_{\text{adm}}}(L_\infty).$$

No Euclidean embedding is part of the definition.

Theorem 4.2 (Regular refinement calibration). Assume the carrier is a regular prefix subtree with the following properties: every level- n carrier cylinder is nonempty and has diameter exactly ρ^n ; every level- n carrier cylinder has exactly $b \geq 2$ pairwise disjoint level- $(n+1)$ carrier-cylinder children; and d_{adm} is the prefix ultrametric, so that for every $x \in L_\infty$ and every radius r with $\rho^{n+1} < r \leq \rho^n$, the ball $B(x, r)$ is contained in the unique level- n cylinder containing x . Then

$$D_H = \frac{\log b}{-\log \rho}.$$

Proof. Put $D = \log b/(-\log \rho)$, so $b\rho^D = 1$. At level n there are exactly b^n carrier cylinders, each of diameter ρ^n . Hence for every $s > D$,

$$\mathcal{H}_{\rho^n}^s(L_\infty) \leq b^n (\rho^n)^s = (b\rho^s)^n \longrightarrow 0,$$

which gives $\dim_H L_\infty \leq D$.

For the reverse inequality, assign Bernoulli cylinder mass

$$\mu([w]) = b^{-|w|}$$

to each carrier prefix cylinder $[w]$. This is consistent because every cylinder has exactly b children, and therefore extends to a Borel probability measure on L_∞ . Fix $x \in L_\infty$ and $0 < r \leq 1$, and choose n with $\rho^{n+1} < r \leq \rho^n$. By the prefix-ball hypothesis, $B(x, r)$ lies in the unique level- n cylinder $[w_n(x)]$. Consequently

$$\mu(B(x, r)) \leq b^{-n} = (\rho^n)^D.$$

Using the strict lower radius inequality (not the upper bound),

$$r > \rho^{n+1} \implies \rho^n < \rho^{-1}r.$$

Therefore

$$\mu(B(x, r)) \leq b^{-n} = (\rho^n)^D \leq \rho^{-D}r^D.$$

The mass-distribution principle now gives $\dim_H L_\infty \geq D$. Combining the two inequalities proves the formula. $\square$

Remark 4.3. No implication “mixed sign $\Rightarrow D_H > 1$ ” is used. Whether D_H is below, equal to, or above one is decided by the actual branching and contraction data.

5. A line can be the limit without being the transport object

Proposition 5.1 (Staircase-hypotenuse separation). Let Γ_n be the rectilinear staircase from $(0, 0)$ to $(1, 1)$ obtained by replacing each diagonal cell by one horizontal and one vertical edge of side $1/n$, and let $L = \{(t, t) : 0 \leq t \leq 1\}$. Then

$$d_H(\Gamma_n, L) \leq \frac{\sqrt{2}}{n} \longrightarrow 0,$$

while

$$\text{length}(\Gamma_n) = 2, \quad \text{length}(L) = \sqrt{2}.$$

Proof. Partition $[0, 1]^2$ into the diagonal squares

$$Q_k = [k/n, (k+1)/n]^2, \quad k = 0, \dots, n-1.$$

The k th staircase step and the diagonal segment $L \cap Q_k$ both lie in Q_k , whose diameter is $\sqrt{2}/n$. Thus every staircase point lies within $\sqrt{2}/n$ of L . Conversely, every point of L lies in some Q_k , and that same square contains the corresponding staircase step, so every diagonal point lies within $\sqrt{2}/n$ of Γ_n . Hence the Hausdorff bound holds in both directions. The staircase has total horizontal length one and total vertical length one, hence length two, while L has Euclidean length $\sqrt{2}$. $\square$

The prime registry gives an exact finite example in which an endpoint observation omits transport variation.

Theorem 5.2 (Finite prime-staircase variation). Put

$$Q(q) = \prod_{p < q} p, \quad S_q(t) = \frac{R(q; Q(q)t)}{\varphi(Q(q))}, \quad 0 \leq t \leq 1.$$

At every finite threshold,

$$S_q(0) = S_q(1) = 0, \quad \text{Var}(S_q) = 2.$$

The normalized state recovers its full one-period remainder by $R(q; x) = \varphi(Q(q))S_q(x/Q(q))$.

Proof. Over one period the affine part $\delta(q)x$ rises by $\varphi(Q(q))$, and the counting staircase contributes exactly $\varphi(Q(q))$ unit jumps. Thus $R(q; \cdot)$ has variation $2\varphi(Q(q))$. Reparameterization and normalization give variation two. The endpoint values follow from the exact count $A(q; Q(q)) = \varphi(Q(q))$. Substitution gives the recovery formula. $\square$

Corollary 5.3 (Endpoint observation does not determine variation). The endpoint pair $(0, 0)$ is shared by each S_q and the zero function, while their total variations are 2 and 0, respectively. Therefore no rule using only that endpoint pair recovers the variation of every finite state.

Proof. A function of the common endpoint pair has one value and cannot equal both variations. $\square$

The complete finite staircase supplies the transport information omitted by its endpoint readout. The successive-prime law propagates that retained state at every admission.

6. Carrier realization and anticonformal normal form

Sign data do not manufacture a Euclidean plane. The transverse realization is additional structure.

Definition 6.1 (Balanced transverse carrier). A carrier realization assigns to $p \in L_\infty$ a tuple

$$\Phi_{\text{car}}(p) = (E_p, g_p, J_p, L_p^+, L_p^-),$$

where (E_p, g_p) is an oriented real two-dimensional inner-product space, $J_p^2 = -I$, and the lines $L_p^\pm$ are transverse. A balanced realization satisfies $L_p^+ \perp L_p^-$, $J_p L_p^+ = L_p^-$, and $g_p(J_p u, J_p v) = g_p(u, v)$; its orientation is the complex orientation of J_p . Proposition 6.3 constructs this metric from arbitrary positive carrier data.

Let C_p act as $+I$ on L_p^+ and $-I$ on L_p^- . Then:

Theorem 6.2 (Canonical anticonformal form). For a balanced carrier choose a unit vector $e \in L_p^+$ and define $f = J_p e \in L_p^-$. In the oriented orthonormal basis (e, f) ,

$$J_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence

$$C_p^2 = I, \quad C_p^* = C_p, \quad C_p J_p = -J_p C_p.$$

With

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

one obtains

$$U^T C_p U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: C.$$

Thus $H_y = \kappa_y C$ with $0 < \kappa_y \leq 1$ satisfies

$$H_y^T H_y = \kappa_y^2 I, \quad H_y J = -J H_y.$$

The displayed off-diagonal signs correspond to the stated oriented basis convention; reversing one basis vector gives the conjugate sign convention and no invariant statement changes.

Proof. Since $f = J_p e$ and $J_p^2 = -I$, $J_p e = f$ and $J_p f = -e$. By definition $C_p e = e$ and $C_p f = -f$. The two displayed matrices follow immediately, as does $C_p J_p = -J_p C_p$. Direct multiplication gives the 45° conjugation. Finally $C^T C = I$, proving the norm and anticonformal identities for H_y . $\square$

Proposition 6.3 (Constructive balanced metric). Let $E = L^+ \oplus L^-$, let $J^2 = -I$ with $JL^+ = L^-$, and let g_0 be any positive inner product. Let $P_\pm$ be the algebraic projections onto the two lines and define

$$g_\perp(u, v) = g_0(P_+ u, P_+ v) + g_0(P_- u, P_- v), \quad g_J(u, v) = \frac{1}{2} \{g_\perp(u, v) + g_\perp(Ju, Jv)\}.$$

Then g_J is positive definite, makes the two sign lines orthogonal, and satisfies $g_J(Ju, Jv) = g_J(u, v)$. With the orientation determined by (e, Je) , every nonzero $e \in L^+$ supplies the balanced carrier after normalization. Thus a compatible carrier metric exists for every such algebraic sign exchange.

Proof. The projection formula is positive definite and separates the lines. Since J exchanges both lines, the two summands defining g_J vanish on a cross-line pair. Positivity follows from the first summand. Finally $J^2 = -I$ interchanges the two summands, so J is an isometry for g_J . Dividing e by its g_J -norm makes both e and Je unit vectors. The construction gives a positive metric adapted to the prescribed sign exchange. $\square$

7. Wake transport, hidden fibers, and exact quotient dynamics

Let a_j be the retained wake entering event j , let B_j be the event response on the wake leg, $P_{L,j+1}$ the longitudinal output projection, and $K_{j+1,j}$ the ordered transfer map. Define

$$M_j = K_{j+1,j} P_{L,j+1} B_j.$$

Then $a_{j+1} = M_j a_j$. If $a_j = c_j + r_j$ with projections $P_j, Q_j = I - P_j$, then

$$c_{j+1} = P_{j+1} M_j c_j + P_{j+1} M_j r_j,$$

$$r_{j+1} = Q_{j+1} M_j c_j + Q_{j+1} M_j r_j.$$

Thus retained residual may re-enter the next canonical channel without becoming a new source.

The observation map need not be injective. In the following linear specialization, let $D_k : W_k \rightarrow Z_k$ and $D_{k+1} : W_{k+1} \rightarrow Z_{k+1}$ be linear deposition maps and let $M_k^w : W_k \rightarrow W_{k+1}$ be linear hidden wake transport. General nonlinear events use the fiber criterion of Theorem 38.1.

Theorem 7.1 (Event-dependent quotient criterion). There exists a unique linear map

$$R_k^w : \text{Ran } D_k \longrightarrow \text{Ran } D_{k+1}$$

satisfying

$$D_{k+1} M_k^w = R_k^w D_k$$

if and only if

$$M_k^w(\text{Ker } D_k) \subseteq \text{Ker } D_{k+1}.$$

Proof. If the kernel condition holds, define $R_k^w(D_k a) = D_{k+1} M_k^w a$. If $D_k a = D_k b$, then $a - b \in \text{Ker } D_k$, hence $M_k^w(a - b) \in \text{Ker } D_{k+1}$ and the definition is independent of representative. Uniqueness on $\text{Ran } D_k$ is immediate. Conversely, if the factorization holds and $a \in \text{Ker } D_k$, then $D_{k+1} M_k^w a = R_k^w D_k a = 0$. $\square$

Corollary 7.2. A non-injective projection can erase wake ancestry while preserving a closed observed successor law. Exact observed evolution therefore does not imply reconstructibility of the hidden finite history.

8. Wallis–Gregory finite transport bound

All quantities in this section are scalars except the retained states in the final theorem. Define

$$\begin{aligned} w_j &= \frac{4(j+1)^2}{(2j+1)(2j+3)}, & W_j &= \prod_{h=0}^j w_h, \\ g_m &= \frac{1}{4m+1} - \frac{1}{4m+3}, & G_m &= \sum_{r=0}^m g_r, \\ \rho_m &= W_{2m} - 2G_m, & \alpha_m &= w_{2m-1}w_{2m}, \end{aligned}$$

and, for $m \geq 1$,

$$f_m = 2G_{m-1}(\alpha_m - 1) - 2g_m, \quad P_m = \prod_{h=1}^m \alpha_h, \quad P_0 = 1.$$

Since $W_0 = 4/3$ and $P_m = W_{2m}/W_0$, the Wallis product limit $W_j \rightarrow \pi/2$ gives

$$P_m \longrightarrow \frac{3\pi}{8}.$$

The Gregory partial sums satisfy $G_m \rightarrow \pi/4$, and therefore $\rho_m \rightarrow 0$.

Lemma 8.1 (Exact remainder recurrence). For every $m \geq 1$,

$$\rho_m = \alpha_m \rho_{m-1} + f_m, \quad \rho_0 = 0.$$

Proof. Using $W_{2m} = \alpha_m W_{2m-2}$ and $G_m = G_{m-1} + g_m$,

$$\begin{aligned} \rho_m &= \alpha_m W_{2m-2} - 2G_{m-1} - 2g_m \\ &= \alpha_m (\rho_{m-1} + 2G_{m-1}) - 2G_{m-1} - 2g_m \\ &= \alpha_m \rho_{m-1} + 2G_{m-1}(\alpha_m - 1) - 2g_m, \end{aligned}$$

which is the stated recurrence. □

Lemma 8.2 (Sign and maximum of the retained remainder). One has

$$f_1 = \frac{8}{525}, \quad f_m < 0 \quad (m \geq 2), \quad 0 < \rho_m \leq \frac{8}{525} \quad (m \geq 1).$$

Moreover, ρ_m decreases strictly for $m \geq 2$.

Proof. Put $x_m = 4m + 1$. Direct algebra gives

$$\alpha_m - 1 = \frac{2x_m^2 + 1}{x_m^2(x_m^2 - 4)}, \quad b_m := \frac{g_m}{\alpha_m - 1} = \frac{2x_m(x_m - 2)}{2x_m^2 + 1}.$$

For $x \geq 5$ define

$$b(x) = \frac{2x(x-2)}{2x^2+1}.$$

Then

$$b'(x) = \frac{4(2x-1)(x+1)}{(2x^2+1)^2} > 0,$$

so b_m is increasing. At $m = 1$, direct evaluation gives

$$\alpha_1 = \frac{192}{175}, \quad f_1 = \frac{8}{525}.$$

At $m = 2$,

$$G_1 = \frac{76}{105} < \frac{126}{163} = b_2.$$

For $m \geq 3$, the finite alternating Gregory upper bound yields

$$G_{m-1} < \frac{\pi}{4} < \frac{263}{315} < \frac{286}{339} = b_3 \leq b_m.$$

Because

$$f_m = 2(\alpha_m - 1)(G_{m-1} - b_m),$$

and $\alpha_m - 1 > 0$, this proves $f_m < 0$ for every $m \geq 2$.

For the retained remainder itself,

$$\rho_m - \rho_{m-1} = (\alpha_m - 1)(W_{2m-2} - 2b_m).$$

At $m = 2$,

$$W_2 = \frac{256}{175} < \frac{252}{163} = 2b_2.$$

For $m \geq 3$, the finite Wallis upper bound gives

$$W_{2m-2} < \frac{\pi}{2} < \frac{526}{315} < \frac{572}{339} = 2b_3 \leq 2b_m.$$

Hence $\rho_m < \rho_{m-1}$ for every $m \geq 2$. Since

$$\rho_1 = f_1 = \frac{8}{525}$$

and $\rho_m \rightarrow 0$, strict decrease to the zero limit implies

$$0 < \rho_m \leq \frac{8}{525}.$$

□

Theorem 8.3 (Exact transported variation). For every $N \geq 1$,

$$\sum_{m=1}^N \frac{f_m}{P_m} = \frac{\rho_N}{P_N},$$

$$\sum_{m=1}^N \frac{|f_m|}{P_m} = \frac{1}{36} - \frac{\rho_N}{P_N},$$

and

$$\sum_{m=1}^N \frac{P_N}{P_m} |f_m| = \frac{P_N}{36} - \rho_N < \frac{\pi}{96}.$$

The supremum of the final finite sum is $\pi/96$.

Proof. Because $P_m = \alpha_m P_{m-1}$, division of the exact recurrence by P_m gives

$$\frac{\rho_m}{P_m} - \frac{\rho_{m-1}}{P_{m-1}} = \frac{f_m}{P_m}.$$

Summing from $m = 1$ to N and using $\rho_0 = 0$ proves

$$\sum_{m=1}^N \frac{f_m}{P_m} = \frac{\rho_N}{P_N}.$$

The first normalized increment is

$$\frac{f_1}{P_1} = \frac{8/525}{192/175} = \frac{1}{72},$$

whereas all later increments are negative by the preceding lemma. Therefore

$$\begin{aligned} \sum_{m=1}^N \frac{|f_m|}{P_m} &= \frac{f_1}{P_1} - \sum_{m=2}^N \frac{f_m}{P_m} \\ &= 2 \frac{f_1}{P_1} - \sum_{m=1}^N \frac{f_m}{P_m} \\ &= \frac{1}{36} - \frac{\rho_N}{P_N}. \end{aligned}$$

Multiplying by P_N gives

$$\sum_{m=1}^N \frac{P_N}{P_m} |f_m| = \frac{P_N}{36} - \rho_N.$$

Since $P_N < 3\pi/8$ at every finite stage and $\rho_N > 0$,

$$\frac{P_N}{36} - \rho_N < \frac{3\pi/8}{36} = \frac{\pi}{96}.$$

Finally $P_N \rightarrow 3\pi/8$ and $\rho_N \rightarrow 0$, so the supremum is exactly $\pi/96$. □

Theorem 8.4 (Uniform retained-state bound). Let X be a normed real vector space with norm $\|\cdot\|_X$. Suppose the complete retained state satisfies

$$x_m = \alpha_m U_m(x_{m-1}) + f_m e_m,$$

where the occupied-state maps $U_m : X \rightarrow X$ need not be globally linear but satisfy, explicitly for every $m \geq 1$,

$$\|U_m(x_{m-1})\|_X \leq \|x_{m-1}\|_X, \quad \boxed{\|e_m\|_X \leq E_w}.$$

Then

$$\|x_N\|_X \leq P_N \|x_0\|_X + E_w \left(\frac{P_N}{36} - \rho_N \right) \leq \frac{3\pi}{8} \|x_0\|_X + \frac{E_w \pi}{96}.$$

Proof. Taking the X -norm in the recurrence and using $\alpha_m > 0$ gives

$$\|x_m\|_X \leq \alpha_m \|U_m(x_{m-1})\|_X + |f_m| \|e_m\|_X \leq \alpha_m \|x_{m-1}\|_X + E_w |f_m|.$$

Divide by $P_m = \alpha_m P_{m-1}$:

$$\frac{\|x_m\|_X}{P_m} \leq \frac{\|x_{m-1}\|_X}{P_{m-1}} + E_w \frac{|f_m|}{P_m}.$$

Summation from $m = 1$ to N gives

$$\|x_N\|_X \leq P_N \|x_0\|_X + E_w P_N \sum_{m=1}^N \frac{|f_m|}{P_m},$$

and the exact transported-variation identity gives the first displayed bound. The limit bound follows from $P_N < 3\pi/8$ and $\rho_N > 0$. $\square$

Remark 8.5. A transport realization uses this estimate through a state map that carries its occupied update into the displayed recurrence and preserves the specified norm. For complete inherited states, Theorem 55.6 derives the remainder budget from the initial state; orthogonal residual projections give the sharp constant $73\pi/192$.

Part II

Prime-word arithmetic and the Riemann zeta function

The finite transport and projection distinction developed in Part I now receives an arithmetic realization. Prime admission determines the event order; the remainder and its Mellin readout provide the corresponding state and observation laws.

The prime-word construction begins before analytic continuation. Prime integers are the primitive letters of the Riemann multiplicative registry, while composite integers are canonical prime-word endpoints with unique prime ancestry. The prime-filtered counting

staircase has an exact native successor law: admitting the next prime q sends $A(q; x)$ to $A(q; x) - A(q; x/q)$ and $R(q; x)$ to $R(q; x) - R(q; x/q)$, while the surviving density is multiplied by $1 - q^{-1}$. Thus subtractive opposition and neighboring multiplicative balance occur in the same finite prime-integer handoff. On the real-variable remainder space $x \geq 0$, the reversible dilations $S(a)f(x) = f(x/a)$ form the free abelian prime-scaling group, while the sieve admissions $H(p) = I - S(p)$ lie in its group algebra; finite products of these admission operators generate every subsequent prime-threshold remainder from any chosen finite starting threshold.

After Mellin readout, the normalized state $K(q; s)$ obeys an exact affine propagation law whose stationary states in the open critical strip are precisely the zeros of $\zeta : K(q^+; s) = K(q; s)$ if and only if $\zeta(s) = 0$. This stationarity characterizes zeros, and its value recovers the canonical exponent by $s = 1 + 1/K$. The exact stationary-state defect below characterizes localization in the canonical coordinate. The fixed-word branch decomposition is separated into the root-inclusive unopened block $T(w, P; s)$ and the root-subtracted proper-descendant residual $\mathcal{R}(w, P; s)$. At a nontrivial zero the actual unopened block vanishes, whereas the analytically continued residual equals $-N^{-\rho}$; this residual is not a surviving nonzero descendant tail.

Gregory, Euler, and Wallis provide independent finite opposition, phase, and neighboring-balance registers before the frozen Gamma-Poisson completion is introduced. At every prime threshold the normalized staircase has total variation 2. Its equal endpoint values therefore record less than its full finite transport.

The arithmetic readout uses the canonical exponent in n^{-s} . Affine coordinates transport this kernel and the zeta function together; they do not move a zero of the unchanged function. The stationary value $K(\rho) = 1/(\rho - 1)$ is faithful to ρ , and its Möbius readout $\Omega(\rho) = \rho/(\rho - 1)$ converts critical-line localization to $|\Omega(\rho)| = 1$. Reflection supplies reciprocal-conjugate pairing. Equality of the paired stationary states is precisely the localization criterion derived below.

9. Starting point and scope

The order of construction matters. Prime integers, their exact successive-threshold sieve update, finite prime words, finite remainders, and logarithmic rate provenance are defined before Mellin, Poisson, Gamma, reflection, or analytic continuation is used as a readout. Section 15 derives the canonical Riemann propagation law directly from admitting the next prime integer; Section 16 identifies the underlying prime-scaling group and proves by finite induction that its group-algebra admission operators generate every subsequent threshold state; Section 17 then shows that nontrivial zeros are exactly stationary normalized states of that finite propagation and gives the corresponding finite product schema. Sections 21,

24, and 25 then serve only as auxiliary nonlinear realizations demonstrating how a scalar endpoint can fail to determine an ordered state; they do not supply the native Riemann update. Section 14 proves the exact finite variation and recoverability of every normalized prime staircase. Section 20 separates the conventionally displayed reflection midpoint from a dynamical displacement and treats the affine family as endowed coordinate readouts rather than arithmetic selectors. Section 28 identifies the intrinsic prime rate $\log p$ and embeds the equilateral registry in the full root-of-unity family. Section 29 combines these facts with the stationary Mellin state and its Möbius circular readout and records the coordinate dependence of the numerical label $1/2$, the unit-circle image, and the complete ordered prime registry.

For $\Re s > 1$ the Riemann zeta function is given simultaneously by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and the Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Outside that half-plane, ζ is defined by analytic continuation; it is meromorphic with a single simple pole at $s = 1$ [1]. The continuation may be written, for $\Re s > 0$ and integer $N \geq 1$, as

$$\zeta(s) = \sum_{k=1}^N k^{-s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx. \quad (9.1)$$

Here $[x]$ denotes the greatest integer not exceeding x , and $\{x\} := x - [x]$. Thus the fractional staircase remainder $\{x\}$ is already an exact component of the analytic continuation, not an informal approximation [1]. The global reflection law may be written [2]

$$\zeta(s) = \chi(s)\zeta(1-s), \quad (9.2)$$

where

$$\chi(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s), \quad (9.3)$$

or equivalently through Riemann's completed function

$$\xi(s) = \frac{1}{2} s(s-1) \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s), \quad \xi(s) = \xi(1-s). \quad (9.4)$$

Accordingly, every conclusion below concerning the conventional label $\Re s = 1/2$ is coordinate-relative to the $C = 1$ Riemann display, an exact compatibility identity, or an explicitly stated pointwise equality criterion. No continuous occupancy of that coordinate, no uniqueness of its geometric representation, and no unproved termwise reflection property is inserted.

10. Historical note: zero value, midpoint readout, and prime counting

In the historical Riemann display, the two numerals that appear in the familiar pair $(1/2, 0)$ occupy different mathematical slots. The equation $\zeta(\rho) = 0$ is a value condition: zero is the value taken by the analytically continued function at the argument ρ . By contrast, $\Re \rho = 1/2$ is a coordinate condition on that argument. Drawing the function as a graph places these numerals side by side, but it does not place them on a common intrinsic metric scale. In particular, the visible separation from 0 to $1/2$ is not a canonical half-unit displacement; it is produced by the chosen graph coordinates. Riemann's use of $s = 1/2 + it$ makes the conventional reflection midpoint visible in the argument coordinate, while the zero remains a function value [3,4].

This distinction also clarifies the historical prime-counting problem. Riemann's 1859 memoir uses analytic continuation and the zeros to obtain formulas for prime counting, but the arithmetic target is not the numeral $1/2$. Prime distribution can be registered through $\pi(x)$, Chebyshev-type counting functions such as $\vartheta(x)$ and $\psi(x)$, or the finite sieve counts used below. The prime number theorem is a counting law, whereas the conventional Riemann-hypothesis formulation adds a zero-localization condition inside one analytic display. Thus uniqueness of $1/2$ as the fixed coordinate of the displayed reflection does not make that numeral an exclusive datum of prime counting [4].

The set-theoretic point is exact. Under an affine relabeling $x = A(s) = a + bs$ with $b \neq 0$, the same reflection involution is conjugated from $J_1(s) = 1 - s$ to $J_C(x) = C - x$, where $C = 2a + b$, and its unique fixed singleton is relabelled from $1/2$ to $C/2$. If the function is written in the relabelled coordinate as $\tilde{\zeta}(x) = \zeta(A^{-1}(x))$, its zero value remains zero while the argument coordinate of the same zero is relabelled. Hence infinitely many graph presentations of the same reflection structure carry a unique fixed readout with different numerical labels. This is coordinate conjugacy of the same set-theoretic structure, not a change of normalization.

The purpose of the construction is to retain the ordered finite structure that is hidden when the Euler product is viewed only as a completed infinite product.

11. Canonical prime words, composite endpoints, and descendant cylinders

Definition 11.1 (canonical prime word). A finite prime word is a nondecreasing sequence

$$w = (q_1, q_2, \dots, q_r), \quad q_1 \leq q_2 \leq \dots \leq q_r,$$

with content

$$N(w) := \prod_{j=1}^r q_j$$

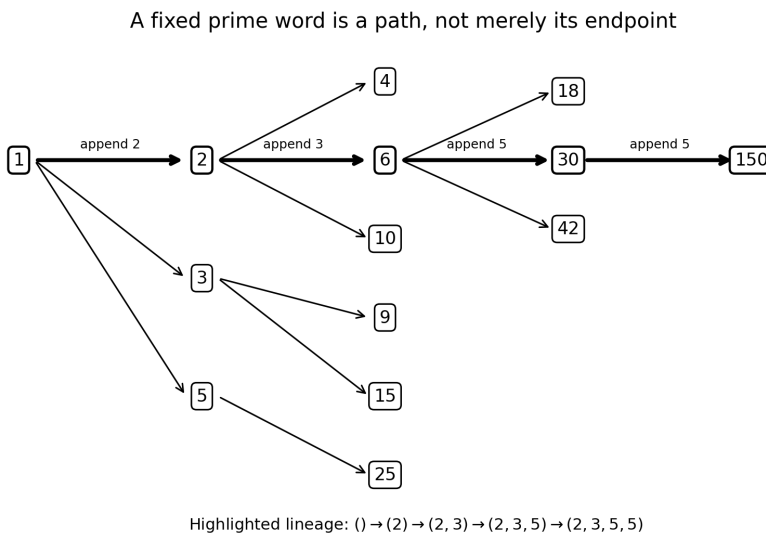
and terminal prime $q(w) := q_r$. The empty word has content $N(\emptyset) = 1$.

A legal extension of a nonempty word appends a prime $p \geq q(w)$:

$$w \mapsto wp = (q_1, \dots, q_r, p).$$

This makes the factorization a path, not merely an endpoint. Figure 1 displays one such path and several competing extensions.

Figure 1. A canonical prime word is retained as an ordered extension path. The highlighted lineage has contents 1, 2, 6, 30, 150.



Lemma 11.2 (unique canonical word). Every integer $n \geq 2$ has exactly one canonical prime word $w(n)$ with $N(w(n)) = n$.

Proof. By the fundamental theorem of arithmetic, n has a unique prime factorization

$$n = \prod_{j=1}^m p_j^{\alpha_j}, \quad p_1 < \dots < p_m,$$

with $\alpha_j \geq 1$. Listing each p_j exactly α_j times in increasing order produces a canonical word. Any other nondecreasing prime word with product n would give a different prime factorization, contradicting uniqueness. $\square$

Proposition 11.3 (composite-letter factorization degeneracy). In the Riemann registry, suppose all prime integers remain primitive letters and a composite integer $c > 1$ is also admitted as an independent primitive letter. Then the word-to-endpoint map is noninjective.

$$\nu((c)) = c = \nu(w(c)), \quad (c) \neq w(c).$$

Proof. Because c is composite, its canonical prime word $w(c)$ has length at least two. The one-letter word (c) is therefore distinct from $w(c)$, while both evaluate to the same numerical endpoint c . Hence endpoint evaluation is noninjective as soon as c is treated as a new primitive in parallel with its prime ancestry.

Corollary 11.4 (composites are blocks, not new Riemann primitives). A composite endpoint may be carried forward as a named transport block only with its canonical prime ancestry attached. Replacing that ancestry by an independent composite primitive creates a second route to the same endpoint and destroys unique primitive provenance. Thus prime integers are the primitive letters of the Riemann word registry, while non-primes are canonical words or ancestry-preserving blocks.

Definition 11.5 (descendant cylinder). For a nonempty word w with terminal prime $q = q(w)$, define

$$\mathcal{D}(w) := \{N(w)m : \text{every prime divisor of } m \text{ is } \geq q\}.$$

The case $m = 1$ is allowed, so the endpoint $N(w)$ belongs to its own cylinder. Every element of $\mathcal{D}(w)$ is obtained by repeatedly extending w with legal primes.

Theorem 11.6 (exact cylinder Dirichlet series). Let w be nonempty, $N = N(w)$ and $q = q(w)$. For $\Re s > 1$,

$$C_w(s) := \sum_{n \in \mathcal{D}(w)} n^{-s} = N^{-s} \prod_{p \geq q} (1 - p^{-s})^{-1}. \quad (11.1)$$

Equivalently, if

$$E_q(s) := \prod_{p < q} (1 - p^{-s}), \quad (11.2)$$

then

$$\boxed{C_w(s) = N^{-s} E_q(s) \zeta(s)}. \quad (11.3)$$

Proof. Every $n \in \mathcal{D}(w)$ has the unique form $n = Nm$, where all prime factors of m are at least q . Hence

$$C_w(s) = N^{-s} \sum_{\substack{m \geq 1 \\ p|m \Rightarrow p \geq q}} m^{-s}.$$

For $\Re s > 1$ this series is absolutely convergent, so unique factorization permits multiplication of the geometric series prime by prime:

$$\sum_{\substack{m \geq 1 \\ p|m \Rightarrow p \geq q}} m^{-s} = \prod_{p \geq q} (1 + p^{-s} + p^{-2s} + \cdots) = \prod_{p \geq q} (1 - p^{-s})^{-1}.$$

Multiplying and dividing the full Euler product by the finitely many factors with $p < q$ gives (11.3). $\square$

12. The finite Euler-product staircase

Let $p_1 = 2 < p_2 = 3 < \cdots$ be the primes and define the finite reciprocal Euler products

$$E_0(s) := 1, \quad E_k(s) := \prod_{j=1}^k (1 - p_j^{-s}). \quad (12.1)$$

Theorem 12.1 (exact prime-step transport). For every complex s and every $k \geq 1$,

$$\boxed{E_{k-1}(s) - E_k(s) = E_{k-1}(s) p_k^{-s}}. \quad (12.2)$$

Consequently, for every finite K ,

$$\boxed{1 - E_K(s) = \sum_{k=1}^K E_{k-1}(s) p_k^{-s}}. \quad (12.3)$$

Proof. From (12.1),

$$E_k(s) = E_{k-1}(s) (1 - p_k^{-s}).$$

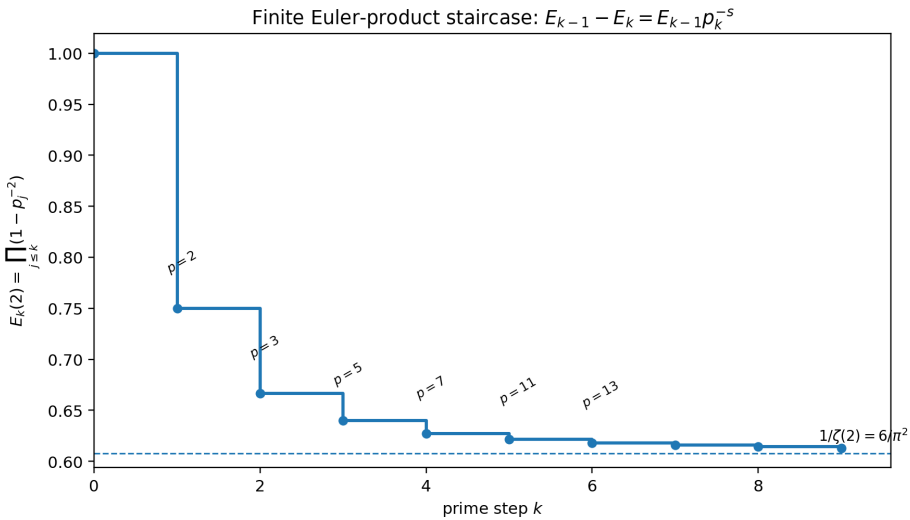
Subtracting from E_{k-1} gives (12.2). Summing (12.2) from $k = 1$ to K telescopes:

$$\sum_{k=1}^K (E_{k-1} - E_k) = E_0 - E_K = 1 - E_K,$$

which is (12.3). No limiting process is used. $\square$

Figure 2 shows the finite recursion at the harmless real value $s = 2$. Each vertical drop is exactly the transported quantity in (12.2).

Figure 2. Finite Euler-product staircase at $s = 2$. Each step is exactly $E_{k-1}(2)p_k^{-2}$.



Proposition 12.2 (finite fixed-word expansion). For every K and every complex s ,

$$E_K(s) = \sum_{J \subseteq \{1, \dots, K\}} (-1)^{|J|} \left(\prod_{j \in J} p_j \right)^{-s}. \quad (12.4)$$

Proof. Expand the finite product $\prod_{j=1}^K (1 - p_j^{-s})$. From each factor choose either 1 or $-p_j^{-s}$. A subset J records exactly those factors from which the second term was chosen, giving sign $(-1)^{|J|}$ and content $\prod_{j \in J} p_j$. $\square$

Thus every finite stage is a finite sum over squarefree prime words, while (12.2) records how the entire existing stage branches when the next prime is admitted.

Corollary 12.3 (first-prime staircase in the convergence half-plane). For $\Re s > 1$,

$$\sum_{k=1}^{\infty} E_{k-1}(s) p_k^{-s} = 1 - \frac{1}{\zeta(s)}. \quad (12.5)$$

Proof. Equation (12.3) holds for every K . In $\Re s > 1$ the Euler product converges and

$$E_K(s) \rightarrow \prod_p (1 - p^{-s}) = \frac{1}{\zeta(s)}.$$

Taking the limit in the finite telescoping identity gives (12.5). $\square$

Proposition 12.4 (the prime limit and analytic continuation are different operations).

Let ρ be any nontrivial zero of ζ . The finite functions $E_K(s)$ cannot converge locally uniformly on any neighborhood of ρ to the meromorphic continuation of $1/\zeta(s)$.

Proof. Each E_K is entire. A locally uniform limit of holomorphic functions on a neighborhood of ρ would be holomorphic there. But $\zeta(\rho) = 0$, so $1/\zeta$ has a pole at ρ (of order equal to the multiplicity of the zero), and is not holomorphic there. Therefore such local uniform convergence is impossible. $\square$

This is the exact mathematical separation between the finite prime staircase and the global analytic continuation. An identity proved by taking $K \rightarrow \infty$ in $\Re s > 1$ cannot simply be evaluated termwise at a zero in the critical strip.

13. Prime-filtered counting staircases and persistent remainders

Fix a prime threshold q and put

$$Q_q := \prod_{p < q} p, \quad \delta_q := \prod_{p < q} \left(1 - \frac{1}{p}\right) = \frac{\varphi(Q_q)}{Q_q}. \quad (13.1)$$

Define the counting staircase

$$A_q(x) := |\{n \in \mathbb{N} : n \leq x, \gcd(n, Q_q) = 1\}|. \quad (13.2)$$

Thus A_q counts integers whose prime divisors are all at least q .

Lemma 13.1 (exact inclusion–exclusion staircase). For every real $x \geq 0$,

$$A_q(x) = \sum_{d|Q_q} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor. \quad (13.3)$$

Proof. For each positive integer n ,

$$\mathbf{1}_{\gcd(n, Q_q)=1} = \sum_{d|\gcd(n, Q_q)} \mu(d).$$

Indeed the right-hand side equals 1 if $\gcd(n, Q_q) = 1$ and 0 otherwise, by the elementary divisor identity $\sum_{d|m} \mu(d) = 0$ for $m > 1$ and equals 1 for $m = 1$. Summing over $n \leq x$ and reversing the finite divisor sum gives

$$A_q(x) = \sum_{d|Q_q} \mu(d) |\{n \in \mathbb{N} : n \leq x, d \mid n\}| = \sum_{d|Q_q} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

□

Definition 13.2 (prime-filtered staircase remainder). Define

$$R_q(x) := \delta_q x - A_q(x). \quad (13.4)$$

Theorem 13.3 (exact fractional representation, periodicity, and boundedness). For all real $x \geq 0$,

$$R_q(x) = \sum_{d|Q_q} \mu(d) \left\{ \frac{x}{d} \right\}. \quad (13.5)$$

Moreover,

$$R_q(x + Q_q) = R_q(x), \quad (13.6)$$

and

$$|R_q(x)| \leq 2^{\omega(Q_q)}, \quad (13.7)$$

where $\omega(Q_q)$ is the number of distinct prime divisors of Q_q .

Proof. Write $[y] = y - \{y\}$ in (13.3):

$$A_q(x) = x \sum_{d|Q_q} \frac{\mu(d)}{d} - \sum_{d|Q_q} \mu(d) \left\{ \frac{x}{d} \right\}.$$

Since Q_q is squarefree,

$$\sum_{d|Q_q} \frac{\mu(d)}{d} = \prod_{p < q} \left(1 - \frac{1}{p}\right) = \delta_q.$$

Rearranging proves (13.5). If $d \mid Q_q$, then Q_q/d is an integer, so

$$\left\{ \frac{x + Q_q}{d} \right\} = \left\{ \frac{x}{d} + \frac{Q_q}{d} \right\} = \left\{ \frac{x}{d} \right\},$$

which proves (13.6). Finally, $|\{x/d\}| < 1$ and the squarefree integer Q_q has $2^{\omega(Q_q)}$ divisors, so the triangle inequality gives (13.7). $\square$

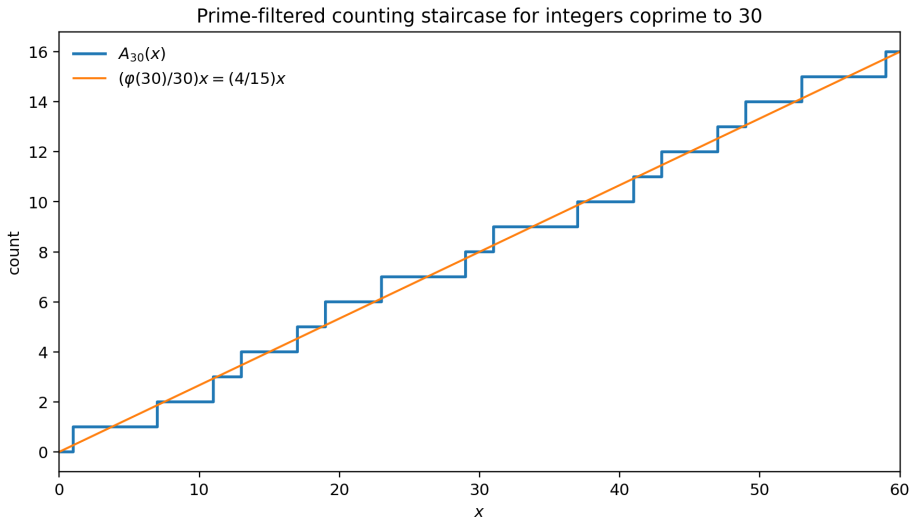
For $q = 2$, the empty product gives $Q_2 = 1$, $\delta_2 = 1$, and

$$A_2(x) = [x], \quad R_2(x) = x - [x] = \{x\}. \quad (13.8)$$

Thus the familiar zeta staircase is the first member of an exact prime-filtered family.

Figure 3 shows A_q for $q = 7$, so $Q_q = 30$ and $\delta_q = \varphi(30)/30 = 4/15$. Figure 4 shows the corresponding periodic remainder.

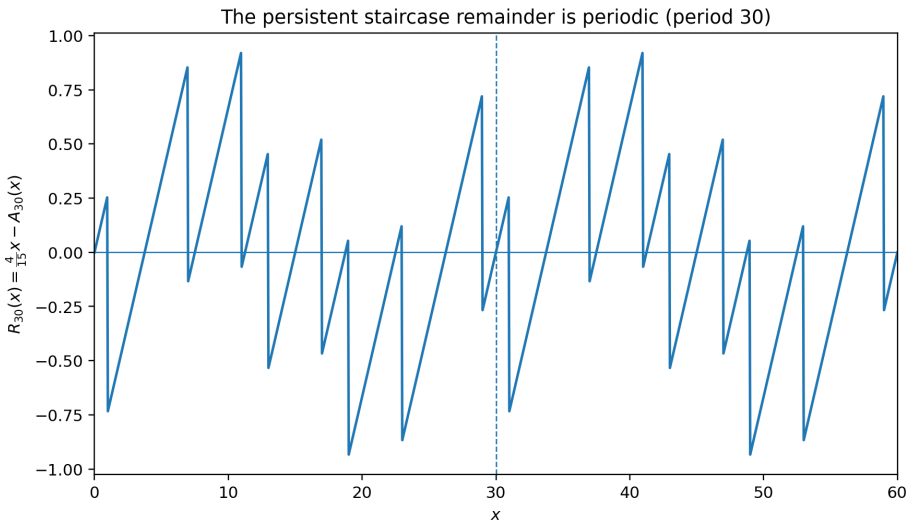
Figure 3. The exact counting staircase $A_7(x)$ for integers coprime to 30, together with its mean-count component $(4/15)x$.



14. Finite staircase transport and exact variation

The arithmetic state is the finite prime-threshold staircase together with its retained remainder. Every prime enters the successive-admission law. Completeness of that ordered registry concerns its prime letters and their finite prefixes; it introduces no continuum of intermediate admissions. The calculations below concern each finite state directly.

Figure 4. The remainder $R_7(x) = (4/15)x - A_7(x)$ repeats exactly with period 30.



Proposition 14.1 (finite staircase and diagonal readout). For an integer $N \geq 1$, let Γ_N be the rectilinear staircase formed by

$$\left(\frac{k}{N}, \frac{k}{N}\right) \longrightarrow \left(\frac{k+1}{N}, \frac{k}{N}\right) \longrightarrow \left(\frac{k+1}{N}, \frac{k+1}{N}\right), \quad 0 \leq k < N,$$

and let $D = \{(t, t) : 0 \leq t \leq 1\}$. Their endpoints agree, and

$$d_H(\Gamma_N, D) \leq \frac{1}{N}, \quad L(\Gamma_N) = 2, \quad L(D) = \sqrt{2}.$$

Proof. The N cells each contain two edges of length $1/N$. A staircase point lies within $1/N$ of the diagonal point with the same varying coordinate. Conversely, if $k/N \leq t \leq (k+1)/N$, then $(t, k/N) \in \Gamma_N$ is within $1/N$ of (t, t) . The diagonal length is $\sqrt{1^2 + 1^2}$. These statements hold at every finite N . $\square$

Theorem 14.2 (exact finite variation of the prime staircase). For the quantities Q_q, δ_q, A_q, R_q of Section 13, define the normalized one-period observation

$$S_q(t) := \frac{R_q(Q_q t)}{\varphi(Q_q)}, \quad 0 \leq t \leq 1.$$

At every finite prime threshold,

$$\boxed{\text{Var}_{[0, Q_q]} R_q = 2\varphi(Q_q), \quad \text{Var}_{[0, 1]} S_q = 2.}$$

In particular, each remainder and each normalized observation is nonzero.

Proof. Between integers, $R_q(x) = \delta_q x - A_q(x)$ rises with slope δ_q . Its continuous rise over a period is $\delta_q Q_q = \varphi(Q_q)$. There are exactly $\varphi(Q_q)$ downward jumps, each of magnitude

one, at the integers coprime to Q_q in that period. Adding the rise and jump magnitudes gives $2\varphi(Q_q)$. The monotone change of variable $x = Q_q t$ preserves variation; division by $\varphi(Q_q)$ gives the second identity. Positive variation excludes the zero function. $\square$

Corollary 14.3 (recovery from the normalized finite state). The normalized observation, with its prime-threshold data retained, recovers the remainder exactly:

$$R_q(x) = \varphi(Q_q)S_q(x/Q_q), \quad 0 \leq x \leq Q_q.$$

Periodicity recovers every other argument. Thus scalar normalization is an exact finite readout with a specified inverse on each period.

Proof. Substitute the definition of S_q and use the period Q_q of R_q . $\square$

Corollary 14.4 (endpoint observation and retained variation). The period endpoints satisfy $R_q(0) = R_q(Q_q) = 0$, while $\text{Var}_{[0, Q_q]} R_q = 2\varphi(Q_q) > 0$. Hence the endpoint pair alone does not determine the transport variation: the zero function has the same endpoints and variation zero.

Proof. The endpoint identities follow from $A_q(0) = 0$ and $A_q(Q_q) = \varphi(Q_q) = \delta_q Q_q$. Apply Theorem 14.2. $\square$

Remark 14.5 (complete prime admission and finite transport). The staircase remains the generating arithmetic object at every admission. Its finite variation and its exact successor law suffice for the construction. A continuous coordinate carrier is an observation space; prime completeness is the inclusion of every prime in the ordered admission process, as proved in Proposition 28.11.

15. Canonical successive-prime propagation before analytic completion

The prime-filtered staircase has a native ordered update that is fixed by the increasing sequence of prime integers itself. No auxiliary state map is required. Let q^+ denote the prime immediately following q . Passing from threshold q to threshold q^+ removes exactly those previously admissible integers that are divisible by the newly admitted prime integer q .

Theorem 15.1 (exact successive-prime sieve handoff). With $Q(q)$, $\delta(q)$, A_q , and R_q as in Section 13,

$$Q(q^+) = qQ(q), \delta(q^+) = (1 - q^{-1})\delta(q)$$

$$A(q^+; x) = A(q; x) - A(q; x/q)$$

$$R(q^+; x) = R(q; x) - R(q; x/q)$$

Proof. Since $Q(q^+) = qQ(q)$ and q does not divide $Q(q)$, an integer survives the q^+ threshold exactly when it survived the q threshold and is not divisible by q . The removed integers are q times integers coprime to $Q(q)$, which gives $A(q^+; x) = A(q; x) - A_q(x/q)$. The density identity follows from the new Euler factor $1 - 1/q$. Substituting both identities into $R(q^+; x) = \delta(q^+)x - A(q^+; x)$ yields the remainder handoff. Every step is finite and exact.

Proposition 15.2 (period-centered opposition of the finite remainder). Assume $q \geq 3$. On one period $[0, Q(q)]$, define the half-jump count $A^\circ(q)$ by assigning half weight at an integer x coprime to $Q(q)$ and the ordinary count elsewhere, and put $R^\circ(q)(x) = \delta(q)x - A^\circ(q)(x)$. Then

$$R^\circ(q; Q(q) - x) = -R^\circ(q; x), 0 \leq x \leq Q(q)$$

Proof. The residue involution $a \rightarrow Q(q) - a$ preserves coprimality and pairs the admissible residues on opposite sides of the period. With half weight at a jump, $A^\circ(q)(Q(q) - x) = \varphi(Q(q)) - A^\circ(q)(x)$. Because $\delta(q)Q(q) = \varphi(Q(q))$, subtraction gives the displayed antisymmetry. The half-jump convention changes values only at isolated points and therefore does not alter any Mellin integral used later.

Corollary 15.3 (finite Gregory-Wallis tension pair). The same prime-integer handoff contains two exact operations: the propagated remainder is the present state minus a scaled copy of its prior state, $R(q^+; x) = R(q; x) - R_q(x/q)$, while the surviving density obeys the neighboring multiplicative balance $\delta(q^+)/\delta(q) = 1 - 1/q$. Section 22 places these operations beside the independent finite Gregory and Wallis registers. The classical constructions are therefore finite prototypes of opposition and neighboring balance, not assumptions used to manufacture the sieve law.

The operational order is now fixed arithmetically: prime admission $\rightarrow$ finite subtractive handoff $\rightarrow$ retained periodic opposition $\rightarrow$ scalar/Mellin readout. Gamma-Poisson is introduced only later as a completed reflection image.

16. Prime-scaling group and finite generation of subsequent states

The one-step successor law of Section 15 already determines every later prime-threshold member. Its algebra is most transparent when the reversible scale action is separated from the noninvertible sieve admission. The scale transformations form a group; the finite exclusion

operators generated from them belong to its group algebra. The canonical increasing-prime sequence then selects an ordered prefix path through these finitely generated states.

Definition 16.1 (prime-scaling group on the real remainder carrier). Let F be the real vector space of real-valued functions $f : [0, \infty) \rightarrow \mathbb{R}$; in particular every finite sieve remainder $R(q; \cdot)$ belongs to F . For every positive rational a define the dilation operator

$$S(a)f(x) := f(x/a)$$

The action is on the real variable x , not on an integer-indexed sequence: for every prime p and every real $x \geq 0$, $S(p)R(q; \cdot)(x) = R(q; x/p)$, including noninteger values of x/p . The fractional evaluations are part of the exact real-valued remainder function and are essential to the finite inclusion-exclusion schema.

Then $S(a)S(b) = S(ab)$, $S(1) = I$, and $S(a)^{-1} = S(a^{-1})$. Hence $G := \{S(a) : a \text{ is a positive rational}\}$ is an abelian group under composition, canonically isomorphic to the multiplicative group of positive rationals. By prime factorization, every a has a finite integer exponent function supported on finitely many primes.

Equivalently, G is the free abelian group generated by the prime dilations $S(p)$; every $S(a)$ is a finite product of integer powers of those generators. This is the operator-level form of unique prime provenance.

Thus the prime integers are not merely labels attached to successive states: on the real carrier F they generate the exact reversible scale architecture. This statement concerns the scale action itself and does not assert that a sieve-admission operator is invertible.

Proposition 16.2 (free prime generators and primitive provenance). Let a be a positive rational and let $m(p) \in \mathbb{Z}$ be its canonical prime-exponent function, nonzero at only finitely many primes. If $S(a) = I$ on F , then $a = 1$ and $m(p) = 0$ for every prime p . Consequently no nontrivial relation exists among the prime dilations, and a composite dilation has no second independent primitive provenance.

Proof. Apply $S(a) = I$ to the test function $f(x) = x$. Since $S(a)f(x) = x/a$, equality with I forces $a = 1$. Unique prime factorization then gives $m(p) = 0$ for every prime p . This is the group-level counterpart of Proposition 11.3: a composite scaling is already the product of its prime scalings, so adjoining it as an independent primitive would duplicate the same factor provenance.

Corollary 16.3 (nonredundancy of successive threshold states). For every prime threshold q , the next state is genuinely new as a real remainder function:

$$\delta(q^+) = (1 - q^{-1})\delta(q) < \delta(q)$$

Thus $R(q^+; \cdot)$ and $R(q; \cdot)$ have different affine slopes on every open interval containing no integer jump. Hence $R(q^+; \cdot) \neq R(q; \cdot)$. Free primitive scaling and strict density change together rule out redundancy of the finite prefix states.

Proposition 16.4 (prime-admission operators in the group algebra). For every prime integer p define

$$H(p) := I - S(p)$$

Then Theorem 15.1 is $R(p^+) = H(p)R(p)$. For primes p and r ,

$$H(p)H(r) = H(r)H(p) = I - S(p) - S(r) + S(pr)$$

Therefore the $H(p)$ are commuting finite elements of the group algebra $\mathbb{Z}[G]$. They need not themselves be invertible group elements. Their product records finite inclusion-exclusion generated by the prime-scaling group.

Proof. Since $S(p)S(r) = S(pr) = S(r)S(p)$, direct expansion of $[I - S(p)][I - S(r)]$ gives the displayed identity. The final statement is exactly the distinction between the group G and its algebra of finite signed combinations.

Theorem 16.5 (finite schema for every subsequent prime-threshold state). Let $q_0 < q_1 < \dots < q_n$ be consecutive prime integers. Then

$$R(q_n) = H(q_{n-1}) \cdots H(q_1)H(q_0)R(q_0)$$

Equivalently, let $\varepsilon(J) = +1$ when $|J|$ is even and $\varepsilon(J) = -1$ when $|J|$ is odd. For every real $x \geq 0$,

$$R(q_n; x) = \sum_{J \subseteq \{0, \dots, n-1\}} \varepsilon(J) R\left(q_0; \frac{x}{\prod_{j \in J} q_j}\right),$$

with the empty product interpreted as 1. Simultaneously,

$$Q(q_n) = Q(q_0)q_0q_1 \cdots q_{n-1}, \delta(q_n) = \delta(q_0)(1 - q_0^{-1})(1 - q_1^{-1}) \cdots (1 - q_{n-1}^{-1})$$

Proof. The case $n = 1$ is Theorem 15.1. Assume the first identity through q_n . Admission of q_n gives $R(q_{n+1}) = H(q_n)R(q_n)$, and substitution of the induction hypothesis appends exactly one further factor $H(q_n)$. This proves the operator product for every finite n . Expanding that finite product gives the subset formula. The Q and δ identities follow by iterating their one-prime updates from Theorem 15.1. No infinite prime product or analytic completion is used.

Corollary 16.6 (schema closure with ordered prefixes). A finite starting state together with the subsequent prime integers determines every later finite threshold state. Although the scale group and the admission operators commute algebraically, the canonical construction retains the prefix chain

$$\mathcal{S}(q_0) \rightarrow \mathcal{S}(q_1) \rightarrow \cdots \rightarrow \mathcal{S}(q_n),$$

because each prefix records which prime thresholds have actually been admitted at that finite stage. The endpoint algebra is commutative, while the construction is an ordered succession of finite events. No interpolating continuous trajectory is required to define that operational motion.

The canonical sieve dynamics are therefore group-generated without being reduced to a static group endpoint: G supplies the reversible real-variable scaling architecture, $\mathbb{Z}[G]$ supplies the finite exclusion operators, and the increasing-prime prefix chain records a nonredundant propagated state history. Mellin transformation in Section 17 reads this already generated orbit into the analytic coordinate.

17. Mellin readout of the finite staircase remainder

This section does not generate the staircase by continuation. The counting remainder of Section 13 is already finite and exact. Mellin transformation is used only to record how that discrete remainder appears in the standard analytic coordinate.

Define the rough-number Dirichlet series

$$Z_q(s) := \sum_{\substack{n \geq 1 \\ \gcd(n, Q_q)=1}} n^{-s}. \quad (17.1)$$

For $\Re s > 1$, Theorem 11.6 with the empty prefactor gives

$$Z_q(s) = E_q(s)\zeta(s). \quad (17.2)$$

Theorem 17.1 (exact Mellin representation and continuation). For $\Re s > 0, s \neq 1$,

$$Z_q(s) = \delta_q \frac{s}{s-1} - s \int_1^\infty R_q(x) x^{-s-1} dx. \quad (17.3)$$

The integral is absolutely convergent in $\Re s > 0$. Consequently (17.3) is a meromorphic continuation of Z_q from $\Re s > 1$ into $\Re s > 0$.

Proof. First suppose $\Re s > 1$. Since $A_q(x) = O(x)$, Stieltjes integration and integration by parts give

$$Z_q(s) = \int_{1-}^{\infty} x^{-s} dA_q(x) = s \int_1^{\infty} A_q(x) x^{-s-1} dx. \quad (17.4)$$

Using $A_q(x) = \delta_q x - R_q(x)$,

$$Z_q(s) = \delta_q s \int_1^{\infty} x^{-s} dx - s \int_1^{\infty} R_q(x) x^{-s-1} dx.$$

For $\Re s > 1$ the first integral equals $1/(s-1)$, proving (17.3) there.

By (13.7), $|R_q(x)| \leq B_q$ for a finite constant B_q . If $\sigma = \Re s > 0$, then

$$\int_1^{\infty} |R_q(x) x^{-s-1}| dx \leq B_q \int_1^{\infty} x^{-\sigma-1} dx = \frac{B_q}{\sigma} < \infty. \quad (17.5)$$

Hence the remainder integral defines a holomorphic function in $\Re s > 0$. The right side of (17.3) is therefore meromorphic there with only the displayed possible pole at $s = 1$, and agrees with Z_q on the nonempty open set $\Re s > 1$. The identity theorem gives the continuation. $\square$

Corollary 17.2 (exact cylinder continuation). Let w have content N and terminal prime q . For $\Re s > 0$, $s \neq 1$,

$$C_w(s) = N^{-s} \left[\delta_q \frac{s}{s-1} - s \int_1^{\infty} R_q(x) x^{-s-1} dx \right]. \quad (17.6)$$

Proof. By (11.3), $C_w(s) = N^{-s} Z_q(s)$ in $\Re s > 1$. Multiply (17.3) by the entire nonzero factor N^{-s} and continue by identity. $\square$

Corollary 17.3 (zero staircase identity). If ρ is a nontrivial zero of ζ , then for every prime threshold q ,

$$\int_1^{\infty} R_q(x) x^{-\rho-1} dx = \frac{\delta_q}{\rho-1}. \quad (17.7)$$

Proof. A nontrivial zero satisfies $0 < \Re \rho < 1$, so $E_q(\rho) \neq 0$ because each factor obeys $|p^{-\rho}| = p^{-\Re \rho} < 1$. Therefore $Z_q(\rho) = E_q(\rho) \zeta(\rho) = 0$. Substitute $s = \rho$ into (17.3). Since $\rho \neq 0$, divide by ρ to obtain (17.7). $\square$

Equation (17.7) is an exact family: the same zero is represented by every prime-filtered periodic staircase remainder.

Definition 17.4 (normalized prime-threshold Mellin state). For $\Re s > 0$ define

$$I(q; s) := \int_1^\infty R(q; x)x^{-s-1}dx, K(q; s) := I(q; s)/\delta(q)$$

Theorem 17.5 (exact Mellin handoff of the canonical sieve state). Let q^+ be the next prime after q . For $\Re s > 0$ and $s \neq 1$,

$$I(q^+; s) = (1 - q^{-s})I(q; s) - \delta(q)q^{-s}(1 - q^{s-1})/(1 - s)$$

Equivalently, the displacement from the fixed state $1/(s - 1)$ obeys

$$K(q^+; s) - 1/(s - 1) = [(1 - q^{-s})/(1 - q^{-1})][K(q; s) - 1/(s - 1)]$$

Proof. Insert the exact handoff into the Mellin integral. In the scaled term set $x = qy$, producing the factor q^{-s} and the lower bound $1/q$. Split the resulting integral at $y = 1$. On $1/q < y < 1$ there is no positive integer counted by $A(q; y)$, so $R(q; y) = \delta(q)y$. The lower interval therefore contributes $\delta(q)[1 - q^{s-1}]/(1 - s)$. Combining that term with the integral from 1 to ∞ gives the displayed formula for $I(q^+; s)$. Using $q^{-s}[1 - q^{s-1}] = q^{-s} - q^{-1}$ and $\delta(q^+) = (1 - q^{-1})\delta(q)$, division by $\delta(q^+)$ gives the normalized affine handoff exactly. No prime-threshold limit is used.

Corollary 17.6 (nontrivial zeros are exactly stationary canonical prime-threshold states). In the open critical strip $0 < \Re s < 1$, for every prime q and its successor q^+ ,

$$K(q^+; s) = K(q; s) \iff K(q; s) = 1/(s - 1) \iff Z(q; s) = 0 \iff \zeta(s) = 0$$

Proof. Let $\alpha_q(s) = (1 - q^{-s})/(1 - q^{-1})$. Stationarity of the affine handoff is equivalent to $[\alpha_q(s) - 1][K(q; s) - 1/(s - 1)] = 0$. Now $\alpha_q(s) = 1$ iff $q^{1-s} = 1$, equivalently $(1 - s) \log q = 2\pi i k$ for some integer k ; every such point has $\Re s = 1$. Hence $\alpha_q(s) \neq 1$ throughout $0 < \Re s < 1$, and stationarity there is exactly $K(q; s) = 1/(s - 1)$. The Mellin identity then gives $Z(q; s) = 0$, and the finite Euler factors relating $Z(q; s)$ to $\zeta(s)$ are nonzero in the open strip, so stationarity is equivalent to $\zeta(s) = 0$.

Remark 17.7 (stationarity and canonical zero localization). Corollary 17.6 proves stationarity at every prime threshold exactly at zeta zeros. The stationary value also determines the canonical exponent, as the next theorem makes explicit. Proposition 29.10 characterizes critical-line localization by the exact stationary-state defect, and Proposition 29.11 gives its reflected one-sided work criterion.

Corollary 17.8 (finite Mellin schema over every subsequent prime block). Define the displacement from the stationary state by

$$E(q; s) := K(q; s) - 1/(s - 1)$$

For consecutive primes $q_0 < q_1 < \cdots < q_n$ and $\Re s > 0, s \neq 1$, repeated application of Theorem 17.5 gives

$$E(q_n; s) = E(q_0; s)[(1 - q_0^{-s})/(1 - q_0^{-1})] \cdots [(1 - q_{n-1}^{-s})/(1 - q_{n-1}^{-1})]$$

Equivalently,

$$K(q_n; s) = 1/(s-1) + [K(q_0; s) - 1/(s-1)][(1 - q_0^{-s})/(1 - q_0^{-1})] \cdots [(1 - q_{n-1}^{-s})/(1 - q_{n-1}^{-1})]$$

At a nontrivial zero ρ , Corollary 17.6 gives $E(q_0; \rho) = 0$, and therefore

$$E(q_n; \rho) = 0, K(q_n; \rho) = 1/(\rho - 1)$$

for every finite n . Thus the zero state is carried unchanged through the entire finite prime-threshold schema rather than reconstructed separately at each threshold.

Proof. Apply the one-step displacement law of Theorem 17.5 successively from q_0 through q_{n-1} . The multipliers accumulate as the displayed finite product. The zero-state statement follows because multiplication preserves the initial value zero.

Remark 17.9 (group-generated orbit versus scalar endpoint). Corollary 17.8 is the Mellin readout of Theorem 16.5. The finite product can be evaluated without retaining its prefixes, but the arithmetic construction itself retains the successive states generated after each prime admission. This is the same endpoint-versus-transport distinction used throughout the manuscript: a completed product may record the final scalar effect while the finite staircase records the operational sequence that produced it.

Theorem 17.10 (Faithful prime characters and stationary-state recovery). In the canonical exponent coordinate the pair $(2^{-s}, 3^{-s})$ determines $s \in \mathbb{C}$ uniquely. At a nontrivial zeta zero the stationary Mellin value gives

$$K(\rho) = \frac{1}{\rho - 1} \neq 0, \quad \rho = 1 + \frac{1}{K(\rho)}.$$

Distinct zeros therefore have distinct stationary Mellin values, even though their zeta function values are both zero.

Proof. Equality of the two characters at s, t gives $(s - t) \log 2 = 2\pi im$ and $(s - t) \log 3 = 2\pi in$ for integers m, n . If either integer is zero then $s = t$. Otherwise $\log 2 / \log 3 = m/n$, implying $2^n = 3^m$, contrary to unique prime factorization, also for signed exponents. The stationary formula is Corollary 17.6; its algebraic inverse proves the rest. $\square$

For $A(s) = a + bs$ with real a, b and $b \neq 0$, a change of display is

$$x = A(s), \quad \zeta_A(x) = \zeta((x - a)/b), \quad p^{-s} = \exp\left(-\frac{x - a}{b} \log p\right).$$

An affine map preserving the unchanged characters 2^{-s} and 3^{-s} at every s must be the identity. In the transported display the reflection center is $a + b/2$ and the phase coefficient with respect to $\text{Im } x$ is $-(\log p)/b$. The intrinsic prime and its logarithmic rate remain the same data, with their coordinate expression transported explicitly.

18. Reflection of a fixed prime-word cylinder

Let $\chi(s)$ be defined by (9.3), so that $\zeta(s) = \chi(s)\zeta(1 - s)$.

Lemma 18.1 (finite Euler factors do not vanish in the critical strip). If $0 < \Re s < 1$, then

$$E_q(s) \neq 0, \quad E_q(1 - s) \neq 0.$$

Proof. For every $p < q$,

$$|p^{-s}| = p^{-\Re s} < 1,$$

so $1 - p^{-s} \neq 0$. Also

$$|p^{-(1-s)}| = p^{-(1-\Re s)} < 1,$$

so $1 - p^{-(1-s)} \neq 0$. Finite products of nonzero factors are nonzero. $\square$

Theorem 18.2 (exact cylinder reflection law). Let w have content N and terminal prime q . In the critical strip, away from the poles of the standard gamma factor,

$$\boxed{C_w(s) = M_w(s)C_w(1 - s)}, \quad (18.1)$$

where

$$\boxed{M_w(s) := \chi(s) N^{1-2s} \frac{E_q(s)}{E_q(1-s)}}. \quad (18.2)$$

Proof. Starting from the continued identity $C_w(s) = N^{-s} E_q(s) \zeta(s)$ and using (9.2),

$$C_w(s) = N^{-s} E_q(s) \chi(s) \zeta(1-s).$$

Since

$$C_w(1-s) = N^{-(1-s)} E_q(1-s) \zeta(1-s) = N^{s-1} E_q(1-s) \zeta(1-s),$$

Lemma 18.1 permits division by $E_q(1-s)$, yielding

$$C_w(s) = \chi(s) N^{1-2s} \frac{E_q(s)}{E_q(1-s)} C_w(1-s).$$

□

Lemma 18.3 (unit modulus at the conventional $C = 1$ midpoint label). If $s = 1/2 + it$ and all factors are regular, then

$$|M_w(s)| = 1. \quad (18.3)$$

Proof. First, $|N^{1-2s}| = N^{1-2\Re s} = 1$. Next, because the coefficients of E_q are real,

$$E_q(1-s) = E_q(\bar{s}) = \overline{E_q(s)},$$

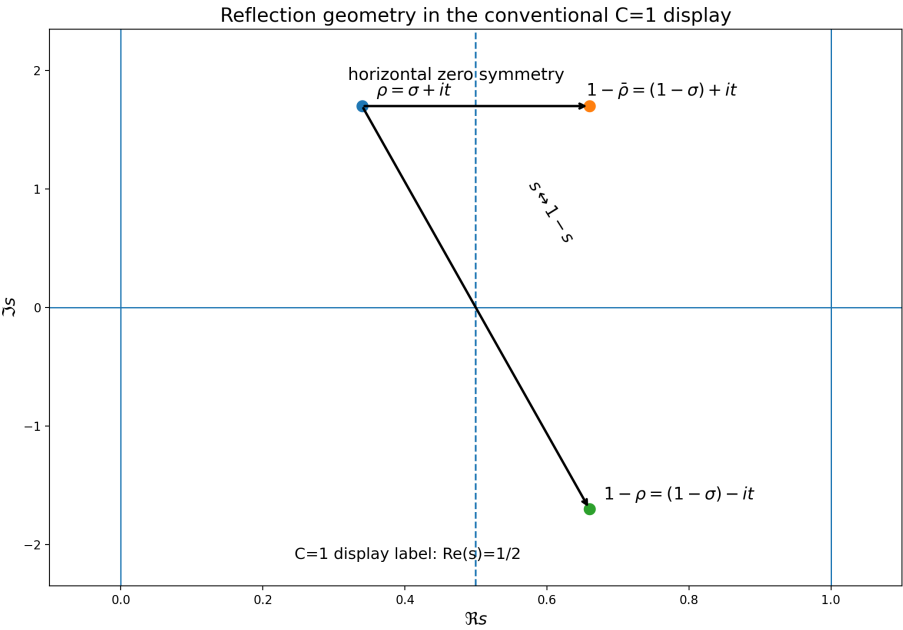
so $|E_q(s)/E_q(1-s)| = 1$. Finally, composing the functional equation with itself gives $\chi(s)\chi(1-s) = 1$ wherever the expressions are regular, and then by analytic identity wherever continued. At the conventional $C = 1$ coordinate $s = 1/2 + it$, one has $1-s = \text{conjugate}(s)$, while $\chi(\text{conjugate}(s)) = \text{conjugate}(\chi(s))$. Hence

$$|\chi(s)|^2 = \chi(s) \overline{\chi(s)} = \chi(s) \chi(1-s) = 1.$$

Multiplying the three unit moduli proves (18.3). □

Figure 5 distinguishes the functional-equation map $s \mapsto 1-s$ from the horizontal zero symmetry obtained after also using complex conjugation.

Figure 5. Reflection geometry in the ambient analytic plane. A zero $\rho = \sigma + it$ produces $1 - \rho = (1 - \sigma) - it$ by the functional equation and $1 - \text{conjugate}(\rho) = (1 - \sigma) + it$ after conjugation. The dashed carrier at the conventional label $\Re s = 1/2$ displays the $C = 1$ reflection coordinate; prime-word phase readouts on that coordinate are defined in Section 28.



19. The zero case: why fixed-word reflection cannot be divided

The fixed-word reflection law is exact. The zero case must therefore be handled exactly as well.

Theorem 19.1 (zero inheritance by every cylinder). Let $0 < \Re \rho < 1$. For every nonempty prime word w ,

$$C_w(\rho) = 0 \quad \Leftrightarrow \quad \zeta(\rho) = 0. \quad (19.1)$$

Proof. In the critical strip,

$$C_w(\rho) = N(w)^{-\rho} E_{q(w)}(\rho) \zeta(\rho).$$

The exponential factor $N(w)^{-\rho}$ is never zero and Lemma 18.1 gives $E_{q(w)}(\rho) \neq 0$. Therefore the product vanishes exactly when $\zeta(\rho)$ does. $\square$

Proposition 19.2 (the reflected zero identity is). $0 = 0$ If ρ is a nontrivial zero, then for every word w ,

$$C_w(\rho) = M_w(\rho) C_w(1 - \rho)$$

reduces to

$$0 = M_w(\rho) 0. \quad (19.2)$$

In particular, (19.2) implies no equation such as $|M_w(\rho)| = 1$.

Proof. Theorem 19.1 gives $C_w(\rho) = 0$. The functional equation gives $\zeta(1 - \rho) = 0$, and Theorem 19.1 applied at $1 - \rho$ gives $C_w(1 - \rho) = 0$. Substitution into (18.1) yields (19.2). Since multiplication of zero by any finite number is zero, the value of $M_w(\rho)$ is not determined by (19.2). $\square$

This closes the hidden division step in the proposed two-word proof. Two equations

$$0 = M_w(\rho) 0, \quad 0 = M_v(\rho) 0$$

cannot be divided to obtain $M_w(\rho) = M_v(\rho)$.

Proposition 19.3 (differentiation does not restore the missing modulus condition).

Suppose ρ is a zero of ζ of multiplicity $m \geq 1$. Then C_w has the same multiplicity and

$$C_w^{(m)}(\rho) = N(w)^{-\rho} E_{q(w)}(\rho) \zeta^{(m)}(\rho). \quad (19.3)$$

Moreover, differentiating the reflection law m times at ρ gives

$$C_w^{(m)}(\rho) = (-1)^m M_w(\rho) C_w^{(m)}(1 - \rho). \quad (19.4)$$

For two words w, v with the same terminal prime q ,

$$\frac{C_w^{(m)}(\rho)}{C_v^{(m)}(\rho)} = \left(\frac{N(w)}{N(v)} \right)^{-\rho}, \quad (19.5)$$

while

$$\frac{C_w^{(m)}(1 - \rho)}{C_v^{(m)}(1 - \rho)} = \left(\frac{N(w)}{N(v)} \right)^{\rho-1}. \quad (19.6)$$

Thus the derivative ratios already carry exactly the same factor $(N(w)/N(v))^{1-2\rho}$ as the reflection multipliers; no new unit-modulus equation appears.

Proof. Write

$$\zeta(s) = (s - \rho)^m g(s), \quad g(\rho) \neq 0.$$

The prefactor $N(w)^{-s} E_{q(w)}(s)$ is holomorphic and nonzero at ρ , so multiplication preserves the order m . The functional equation has a holomorphic nonzero multiplier throughout the open critical strip, so $1 - \rho$ is a zero of the same multiplicity m . The m th derivative at ρ is therefore the prefactor evaluated at ρ times $\zeta^{(m)}(\rho)$, proving (19.3).

For (19.4), differentiate $C_w(s) = M_w(s) C_w(1 - s)$ exactly m times. Every term in which fewer than m derivatives land on $C_w(1 - s)$ contains a derivative of C_w of order strictly less than m evaluated at $1 - \rho$ and vanishes. The surviving term contributes the chain-rule factor $(-1)^m$, giving (19.4).

If w and v share the same terminal prime, the factors $E_q(\rho)$ and $\zeta^{(m)}(\rho)$ cancel in the ratio of (19.3), proving (19.5); applying the same formula at $1 - \rho$ proves (19.6). Dividing (19.5) by (19.6) gives exactly $(N(w)/N(v))^{1-2\rho}$, which is also $M_w(\rho)/M_v(\rho)$ because the common χ and E_q factors cancel. $\square$

20. Affine covariance of the canonically normalized reflection

The zeta function is defined in the exponent coordinate of n^{-s} and continued in that coordinate. Its horizontal reflection is $s \mapsto 1 - \bar{s}$, with fixed locus $\Re s = 1/2$. An affine display transports the function and its kernel as in Theorem 17.10. This covariance preserves the truth or falsity of localization; it does not establish it or supply a different zero of the unchanged function.

Proposition 20.1 (fixed label within the $C = 1$ coordinate display). Let $J(\sigma) = 1 - \sigma$ on the real transverse coordinate. Then J has exactly one fixed point.

$$J(\sigma) = \sigma \Leftrightarrow 1 - \sigma = \sigma \Leftrightarrow \sigma = \frac{1}{2}.$$

Proof. Solving $1 - \sigma = \sigma$ gives $2\sigma = 1$ and hence $\sigma = 1/2$. Uniqueness here means only uniqueness of the fixed label inside this specified affine equation.

Proposition 20.1 therefore identifies only the midpoint label of the conventional $C = 1$ display. It does not assert that every point $1/2 + it$ is occupied, does not prohibit a reflected pair displayed at distinct transverse coordinates, and does not identify coordinate offset with physical or word-state displacement. Occupancy, transport, and projection are separate questions.

Later factors $a^{1/2-s}$ use the midpoint of the canonical reflection. Their unit-modulus locus is that line. Proving unit modulus at every zeta zero is a further assertion, not supplied by the normalization.

Proposition 20.2 (affine conjugacy and portability of the displayed midpoint label). Let $J_1(\sigma) = 1 - \sigma$ and let $A(\sigma) = a + b\sigma$ with $b \neq 0$. Then $A \circ J_1 \circ A^{-1}(x) = (2a + b) - x =: J_C(x)$, where $C = 2a + b$, and $A(1/2) = C/2$. Conversely every real C occurs for an affine choice of a and b . Hence the affine reflection structure has one fixed point, but the numerical labels C and $C/2$ are coordinate data rather than set-wise invariants.

$$J_C(\sigma) = C - \sigma, \quad J_C(\sigma) = \sigma \Leftrightarrow \sigma_* = \frac{C}{2}.$$

Proof. Since $A^{-1}(x) = (x - a)/b$, direct substitution gives $A(1 - A^{-1}(x)) = 2a + b - x$. The fixed point of J_C solves $x = C - x$ and is therefore $C/2$, while $A(1/2) = a + b/2 = C/2$. These are identities of affine coordinate conjugacy; no dynamical assumption enters.

Definition 20.3 (transported transverse readout). For $x = A(s) = a + bs$ put $C = 2a + b$ and $c = C/2$. The line $\Re x = c$ is the image of the canonical reflection line. The displayed function is $\zeta_A(x) = \zeta(A^{-1}x)$ and its characters are $p^{-A^{-1}x}$. Thus c is a display coordinate

for the same marked arithmetic structure. The canonical exponent remains recoverable by Theorem 17.10.

Proposition 20.4 (center-normalized phase neutrality is nonselective). Let c be any endowed readout coordinate (or, algebraically, any real number) and let $a > 1$. Define the center-normalized scalar character $\chi_c(a; s) := a^{c-s}$. Writing $s = \sigma + it$ gives $|\chi_c(a; s)| = a^{c-\sigma}$. Therefore $|\chi_c(a; s)| = 1$ if and only if $\sigma = c$, and on that coordinate $\chi_c(a; c + it) = a^{-it}$. The same pure-phase statement holds at every center inserted into the normalization.

$$\chi_c(a; s) := a^{c-s} = a^{c-\sigma} e^{-it \log a}, \quad |\chi_c(a; s)| = 1 \Leftrightarrow \sigma = c.$$

Corollary 20.5 (centered-character identity). A center-identification argument that inserts c into a^{c-s} and then uses unit modulus of that same character to conclude $\sigma = c$ has merely returned the coordinate inserted into the character, not independently identified a zero locus. In the conventional $C = 1$ display, $c = 1/2$ is the endowed coordinate label; the centered modulus identity therefore supplies no independent pointwise zero-coordinate theorem.

The intrinsic logarithmic rate belongs to the finite multiplicative word. Under $x = a + bs$, its phase coefficient with respect to $\text{Im } x$ is divided by b . Translation alone leaves that coefficient unchanged. Coordinate comparisons transport both reflection and character.

21. Auxiliary metric realizations of word transport

The canonical Riemann propagation is already fixed by Theorem 15.1 and Theorem 17.5. The metric construction in this section has a narrower role: it exhibits possible nonlinear realizations of ordered handoff and proves that a commutative scalar endpoint need not determine a propagated state. These maps are not additional data extracted from the Euler product and are not used to define the successive-prime sieve state. No linear span, Hilbert closure, Gram form, or superposition principle is used.

Definition 21.1 (auxiliary metric word action). Let (X, d) be a metric state space. To every admissible letter a assign a map

$$F(a) : X \rightarrow X$$

For a finite ordered word $w = a_1 a_2 \cdots a_r$ define its action by composition

$$F(w) := F(a_r) \circ \cdots \circ F(a_2) \circ F(a_1)$$

Within this auxiliary realization the word order is retained by composition. Thus $F(b) \circ F(a)$ and $F(a) \circ F(b)$ may differ even when the scalar endpoint satisfies $\text{nu}(ab) = \text{nu}(ba)$.

This demonstrates endpoint underdetermination; it does not replace the canonical increasing-prime threshold order of Section 15.

Definition 21.2 (pointwise handoff distortion). For states X, Y define the extended distortion

$$D(a; X, Y) := \begin{cases} d(F(a)X, F(a)Y)/d(X, Y), & X \neq Y, \\ 1, & X = Y. \end{cases}$$

For distinct states the ratio records local expansion, contraction, or preservation. The value one on a coincident pair is an absorbing convention for subsequent factors: a deterministic map keeps that pair coincident. It is evaluated on the state reaching that step, so it retains the effect of the preceding word.

The scalar zeta weight $\chi(w; s) := \exp(-s \log \nu(w))$ will later be used as a character of this word system. It is a commutative readout of the word, not a replacement for the ordered map $F(w)$.

22. Gregory, Euler, Wallis, and the frozen Gamma image

Before the frozen archimedean completion is introduced, three elementary finite constructions display the staircase architecture directly. None requires analytic continuation of ζ . Gregory supplies the alternating remainder end, Euler supplies a rotating finite-step orbit, and Wallis supplies multiplicative balance. These constructions are used as exact finite witnesses; the later Gamma-Poisson completion is a static image of a different level.

10.1 Gregory: the alternating persistent-remainder staircase

For every integer $N \geq 0$, finite polynomial division gives the exact identity

$$\frac{1}{1+x^2} = \sum_{k=0}^N (-1)^k x^{2k} + (-1)^{N+1} \frac{x^{2N+2}}{1+x^2}.$$

Integrating only this finite identity from 0 to 1 gives

$$\frac{\pi}{4} = G_N + R_N, \quad G_N := \sum_{k=0}^N \frac{(-1)^k}{2k+1}.$$

$$R_N := (-1)^{N+1} \int_0^1 \frac{x^{2N+2}}{1+x^2} dx.$$

Proposition 22.1 (Gregory finite handoff). For every finite N , R_N is nonzero, its sign alternates with N , and the next finite state is obtained by the exact handoff

$$G_{N+1} - G_N = \frac{(-1)^{N+1}}{2N+3}, \quad R_{N+1} = R_N - (G_{N+1} - G_N).$$

Proof. The integrand $x^{2N+2}/(1+x^2)$ is positive on $(0,1)$, so R_N has sign $(-1)^{N+1}$ and cannot vanish at any finite N . The first update is the newly appended Gregory term; subtracting the two exact identities $\pi/4 = G_N + R_N$ and $\pi/4 = G_{N+1} + R_{N+1}$ gives the remainder update.

Thus the Gregory staircase crosses its $\pi/4$ endpoint from alternating sides. The persistent remainder is not a defect to be smoothed away: at each finite stage it is exactly the amount carried into the next handoff.

10.2 Euler: the orbit of finite phase steps

Let $E_N(z) := \sum_{k=0}^N z^k/k!$. The orbit is the ordered finite succession $E_0, E_1, E_2, \dots$ with handoff $h_N := E_{N+1} - E_N$. At $z = i\pi$,

$$h_N = \frac{(i\pi)^{N+1}}{(N+1)!}, \quad \frac{h_{N+1}}{h_N} = \frac{i\pi}{N+2}.$$

Proposition 22.2 (Euler phase handoff). Every finite Euler handoff rotates the scalar phase by $\pi/2$ and rescales its magnitude by $\pi/(N+2)$. The statement is an exact ratio of consecutive finite steps and requires no interpolating trajectory between them.

Proof. Divide $(i\pi)^{N+2}/(N+2)!$ by $(i\pi)^{N+1}/(N+1)!$. The quotient is $i\pi/(N+2)$, whose argument is $\pi/2$ and whose modulus is $\pi/(N+2)$.

Euler is therefore used here as the scalar orbit of the finite steps themselves. No smooth curve is required to exist between one partial state and the next.

10.3 Wallis: the multiplicative balance staircase

Define the finite Wallis product

$$W_N := \prod_{n=1}^N \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{2^{4N}(N!)^4}{(2N)!(2N+1)!}.$$

Proposition 22.3 (Wallis finite balance handoff). The finite states satisfy

$$\frac{W_{N+1}}{W_N} = \frac{[2(N+1)]^2}{(2N+1)(2N+3)}.$$

Proof. All factors common to W_{N+1} and W_N cancel, leaving exactly the newly admitted centered even/odd block. The factorial formula follows by collecting the even and odd products separately.

The classical Wallis limit $W_N \rightarrow \pi/2$ calibrates the endpoint, but only the finite multiplicative handoff is used here. Gregory and Wallis therefore furnish opposite finite π -registers: alternating additive remainder at $\pi/4$ and multiplicative neighboring balance at $\pi/2$, while Euler carries the rotating phase-step orbit between finite states.

Finite balance precedes asymptotic packaging. Gregory, Wallis, and Euler can all be read as finite tension registers: Gregory alternates an exact nonzero remainder across its endpoint, Wallis admits one centered neighboring multiplicative block at each handoff, and Euler advances by an exact complex phase ratio. Their classical infinite limits record terminal scalar values, but the opposition, rescaling, and phase handoff are already present at every finite stage. No limiting interpolation is needed to create that finite tension.

Section 15 now supplies the Riemann-native version of this operational tension. The successive-prime transition combines the subtractive update $R_{q+}(x) = R_q(x) - R_q(x/q)$ with the neighboring density factor $\delta_{q+}/\delta_q = 1 - 1/q$. Gregory and Wallis are not identified term by term with the sieve; rather, their exact finite remainder and balance laws expose the same two operational roles already present in the prime-integer propagation, while Euler supplies the independent phase-step register.

The distinction is important below. A smooth trigonometric, Mellin, theta, or Gamma envelope may package the finite oscillator into a continuous analytic carrier, but the geometry of that carrier is not automatically a uniqueness theorem for the finite registry that generated it. Section 28 makes this explicit for the 120-degree registry by embedding it in the full root-of-unity family.

10.4 Gamma-Poisson: the frozen static image

The archimedean factor is static in a precise sense: it is the Mellin image of one fixed weight. Let $g(x) := \exp(-\pi x^2)$ for $x > 0$ and denote its Mellin transform by $M[g](s)$. The finite Gregory, Euler, Wallis, and prime-word handoffs above are already defined before this frozen weight is introduced.

Proposition 22.4 (the Gamma factor is the frozen Gaussian Mellin weight). For $\Re s > 0$,

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) = 2\mathcal{M}[g](s) = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} dx$$

Proof. The substitution $u = \pi x^2$ transforms the Mellin integral directly into one half of the standard Gamma integral, giving Proposition 22.4 exactly. The Gamma factor is therefore not an approximation to the Gaussian; it is the frozen Gaussian weight after Mellin projection.

Now form the Gaussian-weighted integer staircase

$$\theta(x) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}$$

Poisson summation gives the exact self-dual reflection

$$\theta(x) = x^{-1/2} \theta(1/x)$$

Poisson summation sends the Gaussian-weighted integer lattice to its Fourier-dual lattice, and $\mathbb{Z}$ returns because the integer lattice is self-dual. This is a static reflection statement. The scaling factor in the theta identity supplies the transverse reflection architecture, and Mellin transformation records that architecture in the standard completed zeta factor

$$\Lambda(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad \Lambda(s) = \Lambda(1-s)$$

The role of Gamma here is exhausted by this frozen static completion. It is not asked to propel the prime-word staircase or to generate the persistent remainder. The canonical prime-integer handoff of Section 15 and the independent finite Gregory-Euler-Wallis registers already exist. Gamma-Poisson supplies the completed reflection image in the conventional $C = 1$ coordinate, where the transverse map $\sigma \mapsto 1 - \sigma$ has displayed midpoint label $1/2$. By Section 20, affine conjugacy writes the same fixed-point structure at $C/2$.

The static character of the slot can be checked by changing the weight while leaving ζ and its prime staircase untouched.

Proposition 22.5 (the frozen archimedean weight controls the symmetry type). Define

$$h_2(x) := (4\pi x^2 - 1) e^{-\pi x^2}$$

Under the Fourier convention with kernel $\exp(-2\pi i x \xi)$, direct differentiation of the self-dual Gaussian gives $F[h_2] = -h_2$. Its Mellin transform is

$$\mathcal{M}[h_2](s) = \left(s - \frac{1}{2}\right) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right)$$

Hence the completion built from the same zeta staircase but this frozen weight is

$$\Lambda_H(s) := \mathcal{M}[h_2](s) \zeta(s) = \left(s - \frac{1}{2}\right) \Lambda(s)$$

and therefore

$$\Lambda_H(1-s) = -\Lambda_H(s)$$

The Gaussian produces the symmetric completion; the second Hermite-Gaussian produces an antisymmetric completion containing the coordinate factor $s - 1/2$ in the conventional $C = 1$ display. The prime alphabet and the finite word staircase have not changed. What changed is the frozen archimedean occupant of the reflection slot. Thus Poisson self-duality supplies the static reflection envelope, while the particular frozen weight fixes its symmetry character. Neither operation is used below as a surrogate for the finite word handoff.

23. Nonvanishing descendant staircase observables

The zero identity in Section 19 shows that the cylinder amplitude $C_w(\rho)$ cannot be used as a divisor because it vanishes whenever $\zeta(\rho) = 0$. The staircase remainder itself supplies a different quantity. It is attached to the same fixed word, is obtained from an exact counting identity, and is nonzero at every nontrivial zero.

Definition 23.1 (descendant counting staircase). Let $w = (p_1, \dots, p_r)$ be a nonempty canonical prime word with content

$$N := N(w) = p_1 \cdots p_r$$

and terminal prime $q = p_r$. A descendant of w is obtained by appending primes not smaller than q . Hence every descendant has the unique form Nm with $\gcd(m, Q_q) = 1$. Define

$$A_w(x) := |\{Nm \leq x : \gcd(m, Q_q) = 1\}|. \quad (23.1)$$

Lemma 23.2 (exact dilation of the counting staircase). For every real $x \geq 0$,

$$A_w(x) = A_q(x/N). \quad (23.2)$$

For $x \geq N$ this is equivalently

$$A_w(x) = \frac{\delta_q}{N}x - R_q(x/N). \quad (23.3)$$

Proof. By Definition 23.1, counting descendants $Nm \leq x$ is the same as counting positive integers $m \leq x/N$ with $\gcd(m, Q_q) = 1$. This is exactly $A_q(x/N)$ by (13.2), proving (23.2). If $x \geq N$, then $x/N \geq 1$, and Definition 13.2 gives $A_q(y) = \delta_q y - R_q(y)$. Substituting $y = x/N$ proves (23.3).

Definition 23.3 (fixed-word staircase remainder). For $x \geq N$, set

$$R_w(x) := R_q(x/N). \quad (23.4)$$

Because R_q has period Q_q , R_w has period NQ_q on its natural domain: for $x \geq N$,

$$R_w(x + NQ_q) = R_q(x/N + Q_q) = R_q(x/N) = R_w(x). \quad (23.5)$$

Thus extending a word does not erase the staircase remainder; it dilates its period by the exact word content.

Definition 23.4 (nonvanishing word observable). For $\Re s > 0$, define

$$J_w(s) := \int_N^\infty R_w(x) x^{-s-1} dx. \quad (23.6)$$

Theorem 23.5 (exact Mellin dilation law). If w has content N and terminal prime q , then for $\Re s > 0$,

$$J_w(s) = N^{-s} I_q(s), \quad I_q(s) := \int_1^\infty R_q(u) u^{-s-1} du. \quad (23.7)$$

In particular J_w is holomorphic in $\Re s > 0$.

Proof. Insert (23.4) into (23.6) and substitute $x = Nu$:

$$\begin{aligned} J_w(s) &= \int_N^\infty R_q(x/N) x^{-s-1} dx \\ J_w(s) &= \int_1^\infty R_q(u) (Nu)^{-s-1} N du. \end{aligned}$$

and therefore

$$J_w(s) = N^{-s} \int_1^\infty R_q(u) u^{-s-1} du = N^{-s} I_q(s). \quad (23.8)$$

The function I_q is holomorphic for $\Re s > 0$ by the absolute-convergence estimate (17.5), and multiplication by the entire nonzero factor N^{-s} preserves holomorphy.

Theorem 23.6 (the staircase observable does not vanish at a zeta zero). If ρ is a nontrivial zero of ζ , then for every nonempty prime word w with content N and terminal prime q ,

$$J_w(\rho) = \frac{\delta_q N^{-\rho}}{\rho - 1} \neq 0. \quad (23.9)$$

Proof. Corollary 17.3 gives $I_q(\rho) = \delta_q/(\rho - 1)$. Equation (23.7) therefore gives (23.9). Here $\delta_q > 0$, $N^{-\rho} = e^{-\rho \log N} \neq 0$, and a nontrivial zero has $\rho \neq 1$, so the right side is nonzero.

Corollary 23.7 (same-terminal-word ratio). If w, v have the same terminal prime q and distinct contents N, M , then at every nontrivial zero ρ ,

$$\frac{J_w(\rho)}{J_v(\rho)} = \left(\frac{N}{M} \right)^{-\rho}. \quad (23.10)$$

Proof. Divide the two nonzero identities (23.9). The common factors δ_q and $(\rho - 1)^{-1}$ cancel. Unlike the cylinder equations in Section 19, this division is legitimate because both numerator and denominator are nonzero.

24. Exact Lipschitz distortion cocycle of a word staircase

The following cocycle is an exact statement for any chosen metric realization of the already ordered word. It is auxiliary to the canonical successive-prime update of Section 15: its purpose is to record how history can accumulate through composition without identifying an arbitrary F_a with data forced by prime factorization. Let $w = a_1 \cdots a_r$ and let $X(0), Y(0)$ be distinct states. Define recursively $X(j) = F(a_j)(X(j - 1))$ and $Y(j) = F(a_j)(Y(j - 1))$.

Theorem 24.1 (exact distortion cocycle). With the preceding notation,

$$d(X(r), Y(r))/d(X(0), Y(0)) = D(a_r; X(r - 1), Y(r - 1)) \cdots D(a_1; X(0), Y(0))$$

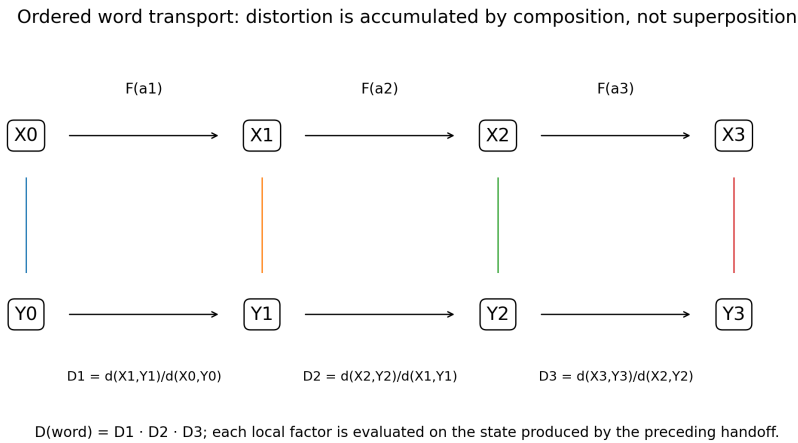
Proof. If no pair merges before the last handoff, every denominator used in the distance ratios is nonzero and cancellation gives the displayed product. Otherwise let j_* be the first index for which $X(j_*) = Y(j_*)$. The factor at j_* is zero with a nonzero denominator. All later states remain coincident, and the defined later factors equal one. Both the product and the final distance ratio are therefore zero. This proves the identity for arbitrary deterministic maps, including noninjective handoffs. $\square$

Corollary 24.2 (Lipschitz bound under composition). If every F_a is Lipschitz, then $\text{Lip}(F(w)) \leq \text{Lip}(F(a_r)) \cdots \text{Lip}(F(a_1))$.

Proof. Apply the Lipschitz estimate successively at each handoff and take the supremum over $X_0 \neq Y_0$.

The distinction from a linear coefficient law is exact. The factor $D(a_j; X(j-1), Y(j-1))$ is generally state-dependent; the same letter can produce a different distortion after a different prefix. The staircase therefore carries history through composition rather than through addition of completed channels.

Figure 6. Lipschitz word cocycle. Two nearby states are propelled through the same ordered letters. The total distortion is the product of the local handoff distortions, each evaluated after the preceding word prefix has acted.



25. Polynomial branching and the scalar zeta shadow

Polynomial maps give one concrete auxiliary nonlinear realization of ordered word action. They are used only to exhibit branching and order-sensitive derivative cocycles that can share a commutative scalar endpoint. The Riemann-native state transition remains the sieve handoff of Section 15. Let each letter a carry a polynomial $P(a)$ and put $z(j) = P(a_j)(z(j-1))$; the ordered composition is $P(w) = P(a_r) \circ \dots \circ P(a_1)$.

Theorem 25.1 (polynomial derivative cocycle). At every point where the derivatives are defined,

$$P(w)'(z(0)) = P(a_r)'(z(r-1)) \cdots P(a_1)'(z(0))$$

Proof. This is the chain rule applied successively to the ordered composition. The evaluation point of each derivative is the state delivered by the preceding handoff.

Thus branching is generated during propagation. Two words may have the same commutative multiplicative endpoint while their polynomial compositions, derivative cocycles, and successor states differ.

Definition 25.2 (scalar character of the word). The multiplicative endpoint defines the scalar character

$$\chi(w; s) := \exp(-s \log \nu(w)), \chi(uv; s) = \chi(u; s)\chi(v; s)$$

After the conventional $C = 1$ coordinate normalization, its scalar shadow is

$$\tilde{\chi}(w; s) := \exp((1/2 - s) \log \nu(w))$$

Proposition 25.3 (transverse neutrality of the scalar shadow). Writing $s = \sigma + it$,

$$|\tilde{\chi}(w; s)| = \exp((1/2 - \sigma) \log \nu(w)).$$

For every nonempty word this modulus equals one exactly when $\sigma = 1/2$. At that coordinate $\tilde{\chi}(w; s) = \exp(-it \log \nu(w))$ is pure phase.

This proposition evaluates scalar balance at the coordinate inserted into the normalization only. It does not determine the nonlinear word map $P(w)$ or its Lipschitz distortion. Those remain properties of the propagated state.

26. Faithful state recovery and the localization criterion

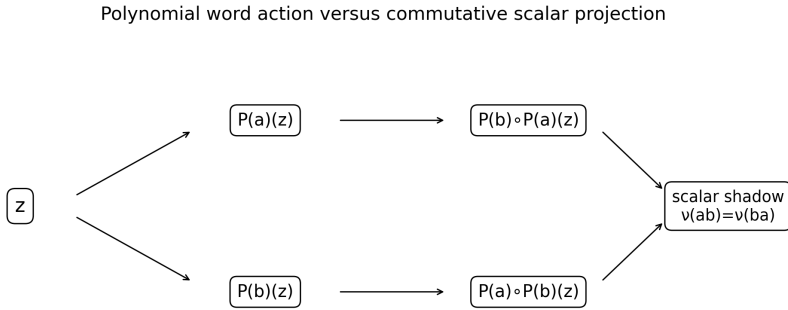
The exact sieve and Mellin handoffs identify stationary zero states without losing the exponent: Theorem 17.10 reconstructs ρ from $K(\rho)$. The nonvanishing word observable also retains its character ratios. Equality of the scalar value zero must be distinguished from equality of these retained states.

Reflection sends the stationary state to the one associated with $1 - \bar{\rho}$. Equality of those two exponents is equivalent to localization and to the defect equality in Proposition 29.10. Affine conjugacy transports this same question. The next section separately retains the root-inclusive unopened block and its root-subtracted continued residual.

27. Root-inclusive unopened blocks and proper-descendant residuals

The fixed-word branch decomposition contains two distinct cutoff objects that must not be conflated. The first is the root-inclusive unopened block left after finitely many child

Figure 7. Polynomial word action versus scalar projection. Ordered compositions can remain distinct even when commutative endpoint evaluation gives the same scalar content. The zeta character records the endpoint shadow; the polynomial staircase retains the handoff history.



The dynamic histories can differ even when their multiplicative endpoint character is identical.

branches are opened. The second is the residual obtained only after the root term has been subtracted from that block. In the absolutely convergent region the residual counts unopened proper descendants; after analytic continuation it remains an exact meromorphic residual, but that counting interpretation is not promoted beyond its domain of absolute convergence.

Let $\nu(p)$ denote the prime immediately following a prime p . Recall

$$E_q(s) = \prod_{r < q} (1 - r^{-s}), \quad Z_q(s) = E_q(s) \zeta(s). \quad (27.1)$$

Lemma 27.1 (one-prime handoff identity). For every prime p ,

$$Z_{\nu(p)}(s) = (1 - p^{-s}) Z_p(s), \quad (27.2)$$

and hence

$$p^{-s} Z_p(s) = Z_p(s) - Z_{\nu(p)}(s). \quad (27.3)$$

These are meromorphic identities in s .

Proof. The primes smaller than $\nu(p)$ are exactly the primes smaller than p together with p . Therefore

$$E_{\nu(p)}(s) = E_p(s) (1 - p^{-s}).$$

Multiplication by $\zeta(s)$ gives (27.2), and rearrangement gives (27.3). Since only a finite Euler factor was introduced, the identities hold wherever the meromorphic continuation of ζ is defined and therefore as meromorphic identities.

Theorem 27.2 (finite prime-branch decomposition). Let $q \leq P$ be primes and let $\nu(P)$ be the next prime after P . Then

$$Z_q(s) = Z_{\nu(P)}(s) + \sum_{q \leq p \leq P} p^{-s} Z_p(s). \quad (27.4)$$

The sum is over primes. If w is a fixed canonical prime word with content N and terminal prime q , then

$$C_w(s) = N^{-s} + \sum_{q \leq p \leq P} C_{wp}(s) + \mathcal{R}_{w,P}(s), \quad (27.5)$$

where

$$\mathcal{R}_{w,P}(s) := N^{-s} (Z_{\nu(P)}(s) - 1). \quad (27.6)$$

Proof. List the consecutive primes from q through P as

$$r_0 = q < r_1 < \cdots < r_m = P,$$

and set $r_{m+1} = \nu(P)$. Summing (27.3) over these primes gives

$$\sum_{q \leq p \leq P} p^{-s} Z_p(s) = \sum_{j=0}^m (Z_{r_j}(s) - Z_{r_{j+1}}(s)) = Z_q(s) - Z_{\nu(P)}(s).$$

Every intermediate term cancels, proving (27.4). Multiplying by N^{-s} gives

$$C_w(s) = N^{-s} Z_{\nu(P)}(s) + \sum_{q \leq p \leq P} N^{-s} p^{-s} Z_p(s).$$

Because appending p gives the word wp with content Np and terminal prime p ,

$$N^{-s} p^{-s} Z_p(s) = C_{wp}(s).$$

Finally split

$$N^{-s} Z_{\nu(P)}(s) = N^{-s} + N^{-s} (Z_{\nu(P)}(s) - 1),$$

which is exactly (27.5)-(27.6).

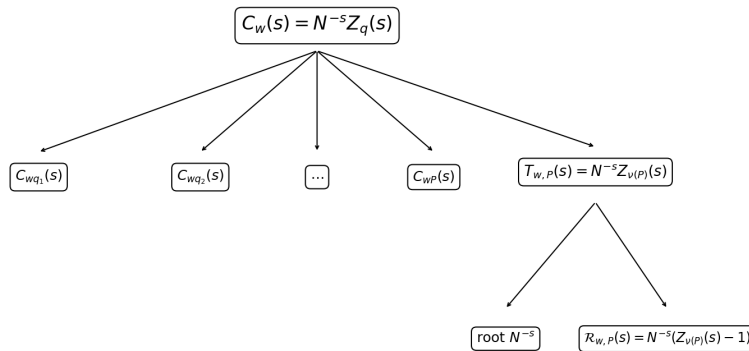
To separate the two objects, write $N = \nu(w)$ and define the root-inclusive unopened block and the root-subtracted residual by

$$T(w, P; s) := N^{-s} Z_{\nu(P)}(s)$$

$$\mathcal{R}(w, P; s) := T(w, P; s) - N^{-s} = N^{-s} [Z_{\nu(P)}(s) - 1]$$

For $\Re s > 1$, $T(w, P)$ has a direct counting meaning as the root together with all unopened descendants whose next prime lies beyond the finite cutoff. After the explicit root split $T(w, P) = N^{-s} + \mathcal{R}(w, P)$, the residual $\mathcal{R}(w, P)$ counts only those unopened proper descendants. Both are exact finite-cutoff objects in the absolutely convergent region; $\mathcal{R}(w, P)$ is not an approximation error, but neither is it identical to the root-inclusive unopened block.

Figure 8. Finite prime-word branching with the root-inclusive unopened block and its root-subtracted residual. The block $T(w, P)$ is what remains unopened after the displayed child branches. Splitting $T(w, P) = N^{-s} + \mathcal{R}(w, P)$ defines the proper-descendant residual. At $a\zeta$ zero the actual unopened block $T(w, P)(\rho)$ is zero, while the analytically continued residual equals $-N^{-\rho}$.



At $\zeta(\rho) = 0$: opened child cylinders vanish and $T_{w,P}(\rho) = 0$; after splitting the block, $\mathcal{R}_{w,P}(\rho) = -N^{-\rho}$.

Theorem 27.3 (continued proper-descendant residual at every nontrivial zero). Let ρ be a nontrivial zero of ζ . For every fixed word w of content N , every terminal prime q , and every finite prime cutoff $P \geq q$,

$$\boxed{\mathcal{R}_{w,P}(\rho) = -N^{-\rho} \neq 0.} \quad (27.7)$$

$$T(w, P; \rho) = 0, \mathcal{R}(w, P; \rho) = -N^{-\rho}$$

Consequently the finite branch decomposition at the zero is exactly

$$\boxed{0 = N^{-\rho} + 0 - N^{-\rho}.} \quad (27.8)$$

Proof. Since $E_r(\rho)$ is a finite product and $\zeta(\rho) = 0$, one has

$$Z_r(\rho) = E_r(\rho)\zeta(\rho) = 0$$

for every finite prime threshold r . In particular $C_w(\rho) = 0$, every finite child cylinder $C_{wp}(\rho) = 0$, and $Z_{\nu(P)}(\rho) = 0$. Substitution into (27.6) gives

$$\mathcal{R}_{w,P}(\rho) = N^{-\rho}(0 - 1) = -N^{-\rho}.$$

Because $N^{-\rho}$ is nonzero, the residual $\mathcal{R}(w, P)(\rho)$ is nonzero. The root-inclusive unopened block is nevertheless exactly zero: the nonzero value appears only after $T(w, P)$ has been decomposed into the explicit root plus the root-subtracted residual.

The value $\mathcal{R}(w, P)(\rho) = -N^{-\rho}$ is exact and independent of the finite cutoff. Its correct interpretation is an analytically continued proper-descendant residual defined after root subtraction. It is not a nonzero descendant tail surviving at the zero, because the actual root-inclusive unopened block $T(w, P)(\rho)$ vanishes.

Corollary 27.4 (persistent family of continued finite-cutoff residuals). Fix a canonical word w and a nontrivial zero ρ . As P runs through successive finite prime cutoffs, the root-inclusive unopened block satisfies $T(w, P)(\rho) = 0$ at every stage, while the root-subtracted residual has the cutoff-independent value $\mathcal{R}(w, P)(\rho) = -\nu(w)^{-\rho}$. The directed family records how the same analytically continued residual appears after different finite branch openings; it is not a family of nonzero root-inclusive tails.

Proof. Theorem 27.2 gives a different finite branch opening at each cutoff P . Theorem 27.3 evaluates the two cutoff objects separately: the unopened block is zero at the ζ zero, whereas subtracting the explicit root leaves the same nonzero residual. No completed $P \rightarrow$ infinity state is used in this finite-cutoff statement.

Theorem 27.5 (the prime-threshold limit and analytic continuation separate the two cutoff objects at a zero). For every s with $\Re s > 1$,

$$\lim_{P \rightarrow \infty} Z_{\nu(P)}(s) = 1, \quad (27.9)$$

where P tends through the primes. At every nontrivial zero ρ ,

$$Z_{\nu(P)}(\rho) = 0$$

for every finite P , and therefore

$$\lim_{P \rightarrow \infty} Z_{\nu(P)}(\rho) = 0. \quad (27.10)$$

$$\lim(P \rightarrow \infty)T(w, P; s) = N^{-s}, \lim(P \rightarrow \infty)\mathcal{R}(w, P; s) = 0 (\Re s > 1)$$

$$T(w, P; \rho) = 0, \mathcal{R}(w, P; \rho) = -N^{-\rho} \text{ (finite } P\text{)}.$$

Thus the two finite-cutoff objects exchange roles under limit versus continuation. In $\Re s > 1$ the root-inclusive block tends to the explicit root N^{-s} while the proper-descendant residual tends to zero. At a nontrivial zero, every finite continued block is zero while every finite continued residual equals $-N^{-\rho}$. This is a noncommutation statement about cutoff and analytic continuation, not evidence for a nonzero unopened descendant block at the zero.

Proof. In $\Re s > 1$, the Euler product converges absolutely and

$$Z_{\nu(P)}(s) = \prod_{p \geq \nu(P)} (1 - p^{-s})^{-1}. \quad (27.11)$$

Write $\sigma = \Re s > 1$. For sufficiently large P , $|p^{-s}| \leq 1/2$ for every $p \geq \nu(P)$. On $|z| \leq 1/2$ one has the elementary bound

$$|\log(1 - z)^{-1}| \leq 2|z|.$$

Hence

$$|\log Z_{\nu(P)}(s)| \leq 2 \sum_{p \geq \nu(P)} p^{-\sigma} \leq 2 \sum_{n \geq \nu(P)} n^{-\sigma} \rightarrow 0.$$

Exponentiating proves (27.9). Equation (27.10) follows from $Z_{\nu(P)}(\rho) = E_{\nu(P)}(\rho)\zeta(\rho) = 0$ for every finite P .

Equivalently, the quantity lost by termwise continuation of the infinite proper-descendant sum is the analytically continued root-subtracted residual. The root-inclusive unopened block itself does not persist at the zero; it vanishes exactly.

Definition 27.6 (conventional $C = 1$ coordinate-normalized residual). At a nontrivial zero define

$$U(w, P; \rho) := N^{1/2} \mathcal{R}(w, P; \rho).$$

By Theorem 27.3 this is independent of the finite cutoff P and equals

$$U(w, P; \rho) = -N^{1/2-\rho}.$$

In the canonical exponent coordinate, requiring the reflected residual product to be independent of word content fixes the normalization exponent to $1/2$, as the calculation below proves. This product condition is weaker than unit modulus of either residual and does not locate a zero. An affine display transports the normalization with its character.

Proof. With normalization N^a , Theorem 27.3 gives residuals $-N^{a-\rho}$ and $-N^{a-(1-\rho)}$. Their product is

$$N^{a-\rho+a-1+\rho} = N^{2a-1}.$$

For all $N > 1$ this is content-independent exactly when $2a - 1 = 0$, namely $a = 1/2$.

Proposition 27.7 (reflected finite-residual identity). Fix a canonical word w of content N and a repeatable legal prime q . Let w_k be obtained by appending k copies of q . At a nontrivial zero ρ and its reflected zero $1 - \rho$, define the normalized finite remainders U_k by Definition 27.6. Then

$$U_k(\rho) = -\left(Nq^k\right)^{1/2-\rho}, \quad U_k(1-\rho) = -\left(Nq^k\right)^{\rho-1/2}$$

and consequently

$$U_k(\rho) U_k(1-\rho) = 1$$

for every finite k .

Proof. Apply Theorem 27.3 to the word w_k at ρ and at $1 - \rho$, then apply the conventional $C = 1$ coordinate normalization of Definition 27.6. The two exponents are opposite, so their product is one for every finite k .

Corollary 27.8 (scalar-character limitation). Proposition 27.7 is an exact reciprocal identity of the scalar shadow. Its dependence on k is geometric because the normalized endpoint character is one-dimensional and multiplicative. No boundedness property of the sequence U_k is used as a closure criterion. On the conventional $C = 1$ displayed coordinate the same scalar shadow is unimodular and oscillatory; Section 20 shows that analogous centered neutrality exists at every chosen center, so this is a compatibility readout rather than a numerical-coordinate selector.

At a nontrivial zero the root-inclusive finite block vanishes, while the root-subtracted continued residual is nonzero and algebraically cancels the explicit root in the split identity. The canonical propagation mechanism is instead the successive-prime sieve law of Sections 15-17. The next section gives exhaustive prime admission and its exact phase readout.

28. Complete prime ancestry and ordered word transport

Section 20 transports the canonical reflection under affine coordinates. Here the word calculations use the original exponent s in n^{-s} and its marked midpoint $1/2$. Their unit-modulus and phase identities are exact on that line; identifying every zeta zero with the line remains separate. Prime integers supply unique ancestry, logarithmic rates, and finite phase handoffs.

Let $\mathcal{A} = \{a_1, a_2, \dots\}$ be any discrete alphabet of real letters with $a_j > 1$. Let $\mathcal{W}_{ord}(\mathcal{A})$ denote the finite ordered words on $\mathcal{A}$, including the empty word $\emptyset$. If

$$w = (b_1, b_2, \dots, b_r),$$

define its multiplicative content by

$$\nu(w) := \prod_{j=1}^r b_j, \quad \nu(\emptyset) := 1.$$

The word itself and its endpoint are different objects. There are two successive projections:

$$\mathcal{W}_{ord}(\mathcal{A}) \rightarrow \mathcal{W}_{com}(\mathcal{A}) \rightarrow \mathbb{R}_{>0},$$

where the first map forgets order and retains only multiplicities, while the second evaluates the formal multiplicative word as the number $\nu(w)$. Thus (a, b) and (b, a) have the same numerical endpoint ab but different prefix histories. For the prime-integer alphabet, unique factorization makes the commutative word-to-endpoint map injective, but it does not restore the ordered history discarded by the first projection.

A finite word carries its own staircase of prefixes. Writing

$$w^{(k)} = (b_1, \dots, b_k), \quad 0 \leq k \leq r,$$

one has the exact chain

$$\emptyset = w^{(0)} \prec w^{(1)} \prec \dots \prec w^{(r)} = w.$$

The passage from w to an admissible extension wa is therefore a discrete handoff, not a motion through an interpolated interval.

Proposition 28.1 (exact word handoff). For $s = \sigma + it$ assign to every finite word the multiplicative weight

$$\omega_s(w) := \nu(w)^{-s}.$$

For every admissible letter $a > 1$,

$$\omega_s(wa) = a^{-s} \omega_s(w).$$

After the conventional $C = 1$ coordinate normalization defined in Definition 27.6,

$$\Theta_s(w) := \nu(w)^{1/2} \omega_s(w) = \nu(w)^{1/2-s},$$

the handoff becomes

$$\boxed{\Theta_s(wa) = a^{1/2-s} \Theta_s(w)}.$$

Consequently,

$$\boxed{\frac{|\Theta_s(wa)|}{|\Theta_s(w)|} = a^{1/2-\sigma}}.$$

Proof. Since $\nu(wa) = a\nu(w)$,

$$\omega_s(wa) = (a\nu(w))^{-s} = a^{-s} \nu(w)^{-s} = a^{-s} \omega_s(w).$$

Multiplying by $(a\nu(w))^{1/2}$ gives

$$\Theta_s(wa) = a^{1/2-s} \nu(w)^{1/2-s} = a^{1/2-s} \Theta_s(w).$$

Taking absolute values and using $|a^{-it}| = 1$ proves the last identity. No property of a beyond $a > 1$ is used. $\square$

Corollary 28.2 (exact unit-modulus criterion). For every admissible letter $a > 1$,

$$\frac{|\Theta_s(wa)|}{|\Theta_s(w)|} = 1 \iff \Re s = \frac{1}{2}.$$

Proof. The ratio is $a^{1/2 - \Re s}$ by Proposition 28.1. Since $a > 1$, it equals one exactly when its exponent vanishes. $\square$

This evaluates the scalar handoff at a supplied s . A localization proof still must establish its left-hand side at every zeta zero.

Proposition 28.3 (radial-phase factorization of the scalar handoff). Write $s = \sigma + it$. For every admissible extension wa , Proposition 28.1 gives

$$\frac{\Theta(wa; s)}{\Theta(w; s)} = \exp\left(\left(\frac{1}{2} - \sigma\right) \log a\right) \exp(-it \log a)$$

The first factor is the transverse scalar distortion and the second is the longitudinal scalar phase. At the endowed $C = 1$ label $\sigma = 1/2$ the radial factor is exactly one and the scalar handoff reduces to pure phase by direct substitution into the chosen normalization.

Proof. Substitute $s = \sigma + it$ into the multiplier of Proposition 28.1 and separate its real and imaginary exponents.

This identity gives the scalar character readout. Pure-phase unimodularity on the $C = 1$ displayed coordinate is an oscillatory property of the multiplicative character; it is not the nonlinear propagation law and no boundedness conclusion is taken from it. The word state remains the ordered handoff map together with its retained remainder and admissible successor data.

The radial-phase factorization now admits an exact finite three-state subregistry when evaluated on the conventional $C = 1$ displayed coordinate.

Proposition 28.4 (prime-rate equilateral phase registry). Let

$$\omega := e^{2\pi i/3}, \quad 1 + \omega + \omega^2 = 0,$$

and, for a prime integer p , define the $C = 1$ -coordinate prime carrier

$$u_p(t) := p^{-it} = e^{-it \log p}.$$

For every real t , the three states

$$\mathcal{E}_p(t) := \{u_p(t), \omega u_p(t), \omega^2 u_p(t)\}$$

form an equilateral triangle centered at the origin. Define the primitive logarithmic phase rate

$$\lambda_p := \log p$$

and the one-third phase step

$$\tau_p := \frac{2\pi}{3 \log p}.$$

Then the finite handoff is exact:

$$u_p(t + \tau_p) = \omega^2 u_p(t),$$

and after three handoffs

$$u_p(t + 3\tau_p) = u_p(t).$$

Thus every prime carries its own primitive logarithmic rate while the same rigid three-state 120° registry is preserved.

Proof. Since $|u_p(t)| = 1$ and $|\omega| = 1$, the three states have equal modulus. Their phase separations are $2\pi/3$, and their centroid is

$$\frac{u_p(t)}{3} (1 + \omega + \omega^2) = 0.$$

Hence they form an equilateral triangle centered at zero. Moreover,

$$\tau_p \log p = \frac{2\pi}{3},$$

so

$$\frac{u_p(t + \tau_p)}{u_p(t)} = e^{-i\tau_p \log p} = e^{-2\pi i/3} = \omega^2.$$

Cubing this multiplier gives 1, proving the three-handoff return. No analytic continuation or limiting operation is used. $\square$

Corollary 28.5 (unique rate provenance of composite endpoints). Let

$$n = \prod_p p^{v_p(n)}$$

be the canonical prime factorization of a positive integer. Then its $C = 1$ -coordinate carrier factors exactly as

$$n^{-it} = \prod_p (p^{-it})^{v_p(n)} = \exp \left(-it \sum_p v_p(n) \log p \right),$$

and therefore its logarithmic rate is

$$\lambda_n = \log n = \sum_p v_p(n) \lambda_p.$$

A composite contributes no new independent rate. If a composite c is introduced as an independent primitive beside its prime factors, the one-letter word (c) and the canonical prime word $w(c)$ have the same endpoint and the same accumulated rate but distinct primitive histories. Proposition 11.3 is therefore kinematic as well as algebraic: independent composite letters create nonunique rate provenance.

Definition 28.6 (finite aggregate phase projection). For a finite admissible word family $\mathcal{W}$ and coefficients $a_w \in \mathbb{C}$, define

$$\Pi_{\mathcal{W},a}(t) := \sum_{w \in \mathcal{W}} a_w e^{-it \log \nu(w)}.$$

For a prime-only cutoff this specializes to

$$\Pi_{P,a}(t) = \sum_{p \leq P} a_p p^{-it}.$$

The 120-degree equilateral registry is an exact property of each individual prime carrier, but it is neither a unique cyclic balance law nor a prediction of a particular smooth envelope. Any cardioid, deltoid, or other rounded envelope obtained from the finite aggregate projection belongs to the explicitly chosen word and coefficient families and must be computed or bounded from those data. The finite registry, the finite aggregate, and any limiting envelope are distinct objects in exactly the staircase-hypotenuse sense of Section 14.

Proposition 28.7 (root-of-unity cyclic balance family). Let $m \geq 2$ and $\omega_m := \exp(2\pi i/m)$. For every prime integer p and real t , the m phase states $u_p(t)\omega_m^j$, $j = 0, \dots, m-1$, have exact zero centroid. With the primitive rate $\Omega_p = -\log p$, the elementary m -fold handoff time is $\tau_{p,m} := 2\pi/(m \log p)$.

$$\omega_m := e^{2\pi i/m}, \quad \sum_{j=0}^{m-1} \omega_m^j = 0.$$

$$\tau_{p,m} := \frac{2\pi}{m \log p}, \quad u_p(t + \tau_{p,m}) = \omega_m^{-1} u_p(t).$$

Proof. The centroid identity is the finite geometric sum $(1 - \omega_m^m)/(1 - \omega_m) = 0$ for $\omega_m \neq 1$. The handoff follows because $\tau_{p,m} \log p = 2\pi/m$. After m handoffs the multiplier is $\omega_m^{-m} = 1$. Every identity is finite and exact.

Corollary 28.8 (cyclic balance is not a uniqueness selector). Root-of-unity cancellation alone does not select $m = 3$. The equilateral registry is the smallest non-collinear member of the family, but $m = 2, 4, 5, \dots$ give equally exact cyclic cancellation laws. Consequently, any claim that the three-state registry is intrinsically selected by the Riemann prime-word staircase requires an additional arithmetic selection theorem; it cannot follow from harmonic geometry alone.

The equilateral triad therefore supplies one exact three-state subregistry inside the conventional $C = 1$ phase display, while Proposition 28.7 records the larger cyclic class to which it belongs. Every prime enters the ordered registry through its exact admission law; the phase display records the corresponding character of that finite event.

The fixed-word scalar law is now connected to the continued finite-cutoff residual after the root split, while the canonical Riemann propagation remains the successive-prime sieve state of Sections 15-17.

Theorem 28.9 (the continued finite-cutoff residual has the same scalar handoff). Let ρ be a nontrivial zero of the Riemann zeta function, let w be a canonical prime word of content N , and let P be any finite next-prime cutoff. By Theorem 27.3 the continued proper-descendant residual satisfies

$$\mathcal{R}_{w,P}(\rho) = -N^{-\rho}.$$

Hence its conventional $C = 1$ -coordinate normalized value is exactly

$$U(w, P; \rho) = N^{1/2} \mathcal{R}(w, P; \rho) = -\Theta_\rho(w) = -N^{1/2-\rho}$$

For a legal repeated- q extension w_k , then

$$\frac{N_{k+1}^{1/2} \mathcal{R}_{w_{k+1}, P}(\rho)}{N_k^{1/2} \mathcal{R}_{w_k, P}(\rho)} = q^{1/2-\rho},$$

where $N_k = Nq^k$.

Proof. The first identity is Theorem 27.3, and Definition 27.6 gives the normalized identity. Passing from w_k to w_{k+1} multiplies the word content by q , so division of consecutive normalized residuals gives exactly the displayed scalar multiplier.

Thus the word multiplier in Proposition 28.1 is also the multiplier of the continued finite-cutoff residual after the root split. This remains a scalar fixed-word identity. The Riemann-native ordered propagation is the successive-prime state update of Sections 15-17, while the local word character and the residual factor are readouts attached to a chosen word.

The prime-integer alphabet supplies unique primitive ancestry and, through its characters, a faithful exponent readout. The canonical reflection fixes its midpoint at $1/2$; changing display transports both midpoint and kernel. An independent composite primitive duplicates ancestry, whereas an ancestry-preserving composite block remains valid. Other alphabets can exhibit the formal handoff without inheriting this arithmetic registry.

Figure 9 separates these levels explicitly. The staircase lives in successive word extensions. Commutative factorization and numerical endpoint evaluation are later projections of that history.

Definition 28.10 (ordered prime-admission registry). Write $p_1 = 2 < p_2 < p_3 < \dots$ for the primes, and put

$$\mathcal{P}_m = \{p_1, \dots, p_m\}, \quad Q_m = \prod_{j=1}^m p_j, \quad Q_0 = 1.$$

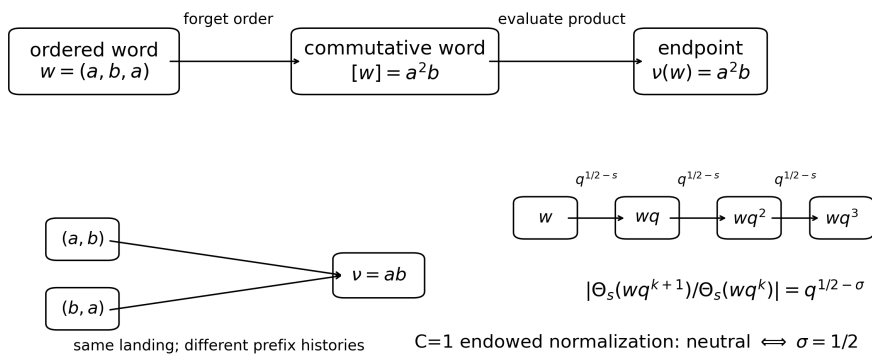
The event from threshold p_m to p_{m+1} admits the letter p_m and carries the complete finite remainder through the exact handoff of Theorem 15.1.

Proposition 28.11 (exhaustive prime admission). Every prime is admitted at a finite event, exactly once, and

$$\bigcup_{m \geq 0} \mathcal{P}_m = \mathcal{P}.$$

Every positive integer endpoint has a finite canonical prime word whose letters all belong to some $\mathcal{P}_m$.

Figure 9. Auxiliary ordered-word realization versus scalar shadow. The diagram illustrates order sensitivity that a commutative endpoint cannot recover; it is not the canonical prime-sieve ordering. The Riemann-native successive-prime state transition is Theorem 15.1. Along a chosen repeated-letter word, the normalized scalar character still carries the exact multiplier of Proposition 28.1, but that auxiliary realization is not used as the zero-state closure.



The staircase lives in successive word extensions; endpoint evaluation is a later projection.

Proof. For a prime p , its rank $m = \#\{r \in \mathcal{P} : r \leq p\}$ is finite, and $p = p_m$. Strict ordering gives unique admission. Conversely every admitted letter is prime. Unique factorization supplies the finite canonical word of any integer; the largest index of its prime factors places all of its letters in one finite prefix. $\square$

Completeness here is arithmetic: every prime is accounted for. The registry consists of these ordered admissions and their retained states. The definition specifies discrete admissions and the exact state passed between them.

Proposition 28.12 (phase readout of a prime extension). For a word w of content N and a legal prime extension wp , the displayed coordinate $s = 1/2 + it$ gives

$$\Theta_{1/2+it}(w) = e^{-it \log N}, \quad \Theta_{1/2+it}(wp) = e^{-it \log p} \Theta_{1/2+it}(w).$$

Proof. Use $\nu(wp) = p\nu(w)$ in Proposition 28.1. $\square$

Corollary 28.13 (prime completeness and coordinate observation). The same complete prime alphabet supplies the factors p^{-it} in every such phase display. The variable t parameterizes the observation; the ordered prime events and their remainder handoffs define the registry being observed.

Proof. Proposition 28.11 supplies every prime letter, and Proposition 28.12 gives its exact phase factor. $\square$

For visualization, let $C(w, k)$ denote a displayed support of the finite word state w at event k . Figure 10 arranges four such observations on a chosen complex-coordinate axis. The colored supports record word observations, and the arrows record their successive handoffs. The arithmetic state is generated by the exhaustive prime-admission law of Proposition 28.11.

29. Stationary-state recovery and the reflection criterion

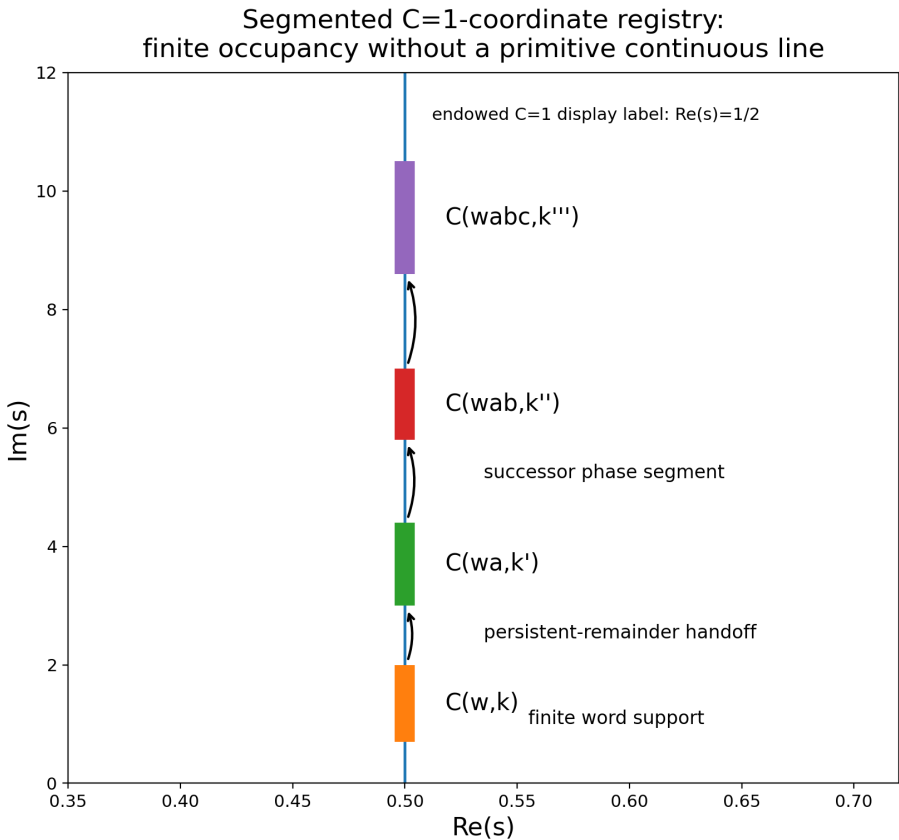
The prime-threshold handoff, finite Mellin schema, and canonical exponent recovery are proved. This section expresses localization as a scalar defect of the retained stationary state. Coordinate covariance is explicit, so a transformed display is distinguished from the unchanged arithmetic function.

Definition 29.1 (stationary zero state and Möbius readout). Let ρ be a nontrivial ζ zero. Corollary 17.6 gives the same normalized stationary value for every finite prime threshold q :

$$K(\rho) = 1/(\rho - 1), \Omega(\rho) = 1 + K(\rho) = \rho/(\rho - 1)$$

The value $K(\rho)$ is generated from the prime-filtered Mellin state and is independent of the

Figure 10. Coordinate display of successive finite word observations. The colored supports and arrows represent retained word states and their handoffs. The vertical axis is a chosen coordinate carrier for these observations; prime admission is governed by the complete ordered registry. Every prime is admitted, and the finite remainder passes through each successive event.



cutoff q . Ω is only a Möbius readout of that already stationary state; no midpoint has been inserted into its definition.

Proposition 29.2 (stationarity and distinguishable reflected states). Every nontrivial zero is stationary under every prime handoff. If ρ and $1 - \bar{\rho}$ differ, their stationary values differ.

Proof. Apply Corollary 17.6 and the inverse formula of Theorem 17.10. The handoff preserves each state; it does not identify them merely because both have zeta value zero. $\square$

Proposition 29.3 (prime-integer ancestry is already closed). Within the Riemann integer registry, every positive integer endpoint has one canonical ancestry in prime integers, and no independent composite primitive can be added without producing endpoint degeneracy.

Proof. Lemma 11.2 gives existence and uniqueness of the canonical prime word by the fundamental theorem of arithmetic. Proposition 11.3 shows that a composite primitive would create a second distinct word with the same endpoint. Thus the primitive arithmetic side is finite and exact before continuation: the primitive letters are prime integers, and the successive-prime sieve law acts on data with unique factor provenance.

Proposition 29.4 (centered-character modulus). Let c be real, $a > 1$, and $s = \sigma + it$. Then $|a^{c-s}| = a^{c-\sigma}$, so $|a^{c-s}| = 1$ exactly when $\sigma = c$. An argument that first installs c in the centered character and then uses unit modulus of that same character to conclude $\sigma = c$ has returned the center built into the representation. It is a compatibility identity, not an independent pointwise zero-identification theorem.

Proof. Since $a > 1$, the positive factor $a^{c-\sigma}$ equals one exactly at $\sigma = c$. The same calculation holds for every c .

Proposition 29.5 (the limited center-erasure statement). An observation retaining only finite counting data and the recentered phase $N^{c-(c+it)} = N^{-it}$ has erased c and cannot recover it. The full canonical characters and stationary Mellin value are not this reduced observation.

Proof. The retained expression is independent of c . The faithful recovery formulas of Theorem 17.10 use different, uncentered data. $\square$

Corollary 29.6 (coordinate covariance preserves localization). For $A(s) = a + bs$, $b \neq 0$, localization of every nontrivial zero of $\zeta(s)$ on $\Re s = 1/2$ is equivalent to localization of the corresponding zeros of $\zeta_A(x)$ on $\Re x = a + b/2$.

Proof. Both zeros and line are carried bijectively by A . The unchanged equation $\zeta(x) = 0$ is not substituted for $\zeta_A(x) = 0$ in this comparison. $\square$

Proposition 29.7 (circular family of stationary-state readouts). Let $\rho = \sigma + it$ be any point with $0 < \sigma < 1$ and set $\Omega = \rho/(\rho - 1) = x + iy$. Then the vertical line $\Re \rho = \sigma$ maps to the circle

$$(1 - \sigma)(x^2 + y^2) + (2\sigma - 1)x - \sigma = 0, \Omega = x + iy, \Re \rho = \sigma$$

whose center and radius are

$$c(\sigma) = (1 - 2\sigma)/[2(1 - \sigma)], r(\sigma) = 1/[2(1 - \sigma)]$$

Under the horizontal Riemann reflection $\rho \longrightarrow 1 - \text{conjugate}(\rho)$,

$$\Omega(1 - \bar{\rho}) = 1/\bar{\Omega}(\rho)$$

and in the conventional $C = 1$ coordinate

$$\Re \rho = 1/2 \iff |\Omega(\rho)| = 1$$

Proof. Solving $\rho = \Omega/(\Omega - 1)$ and taking real parts gives the displayed circle equation; completing the square gives its center and radius. Direct substitution into the reflected point gives reciprocal conjugation. The unit-circle equivalence follows from $|\rho| = |\rho - 1|$. Thus every transverse vertical carrier has a circular Möbius image; the conventionally labeled line $\Re \rho = 1/2$ is the member whose displayed circle is centered at the origin with radius one; the unit-circle statement is therefore a property of this displayed coordinate.

Proposition 29.8 (affine transport of the circular readout). Let $A(s) = a + bs$ with $a, b \in \mathbb{R}$ and $b \neq 0$, and set $C = 2a + b$. The transported readout is

$$\Omega_A(x) := \Omega(A^{-1}x) = \frac{x - a}{x - a - b}.$$

For finite $x \neq a + b$,

$$|\Omega_A(x)| = 1 \iff \Re x = a + \frac{b}{2} = \frac{C}{2}.$$

Moreover $\Omega_A(C - \bar{x}) = 1/\overline{\Omega_A(x)}$ wherever the reciprocal is finite, and the identity extends to the Riemann sphere. This construction includes $C = 0$; the two marked points a and $a + b$ remain distinct.

Proof. Equality of moduli is $|x - a| = |x - a - b|$. Subtracting the squared distances gives $2b(\Re x - a) = b^2$. Since $b \neq 0$, the asserted line follows. Substitution into $C - \bar{x}$ gives the reciprocal-conjugate formula. Thus the normalization is transported with the coordinate. The special formula $s/(s - C)$ is valid only for $a = 0, b = C \neq 0$; it is not the general affine family. $\square$

Corollary 29.9 (exact pointwise equality criterion in the $C = 1$ display). For a nontrivial ζ zero ρ , the coordinate statement $\Re \rho = 1/2$ in the conventional $C = 1$ display is equivalent to the nonlinear stationary-state condition

$$|\Omega(\rho)| = 1 \iff K(\rho)\overline{K}(\rho) + K(\rho) + \overline{K}(\rho) = 0.$$

This equality concerns the faithful stationary value in the canonical coordinate. Reciprocal-conjugate pairing alone does not force fixity.

Proposition 29.10 (Exact stationary-state localization defect). At every nontrivial zero $\rho = \sigma + it$, put

$$\mathcal{D}(\rho) = |1 + K(\rho)|^2 - 1 = |K(\rho)|^2 + K(\rho) + \overline{K}(\rho).$$

Then

$$\boxed{\mathcal{D}(\rho) = \frac{2\sigma - 1}{|\rho - 1|^2}.$$

The Riemann hypothesis is equivalent to $\mathcal{D}(\rho) = 0$ at every nontrivial zero. The defect is independent of the prime threshold.

Proof. Use $1 + K(\rho) = \rho/(\rho - 1)$ and $|\rho|^2 - |\rho - 1|^2 = 2\sigma - 1$. The denominator is positive. The stationary value, hence its defect, is unchanged at every threshold. $\square$

This gives the localization criterion in terms of the retained stationary state. Proposition 29.11 converts this equality into a one-sided work estimate for the reflected stationary readouts.

Proposition 29.11 (One-sided work criterion for stationary localization). For a nontrivial zero ρ , write $\rho^* = 1 - \bar{\rho}$ and $\Omega(\rho) = 1 + K(\rho)$. Then

$$\mathcal{D}(\rho^*) = -\frac{\mathcal{D}(\rho)}{1 + \mathcal{D}(\rho)}. \quad (29.1)$$

Consequently, proving $\mathcal{D}(\rho) \leq 0$ for every nontrivial zero is sufficient for $\mathcal{D}(\rho) = 0$ at every such zero.

In real coordinates, define the anticonformal stationary readout

$$H_\rho = \begin{pmatrix} \Re \Omega(\rho) & \Im \Omega(\rho) \\ \Im \Omega(\rho) & -\Re \Omega(\rho) \end{pmatrix}.$$

Its exact work matrix is

$$H_\rho^T H_\rho - I = \mathcal{D}(\rho)I. \quad (29.2)$$

Thus the one-sided criterion is equivalently the nonexpansiveness of this specified stationary readout at every zero.

Proof. Reflection preserves the nontrivial zero set and gives $\Omega(\rho^*) = 1/\overline{\Omega(\rho)}$. The quantity $1 + \mathcal{D}(\rho) = |\Omega(\rho)|^2$ is positive. Taking squared moduli and subtracting one proves (29.1). If both defects are nonpositive, that identity makes each zero. Direct multiplication proves (29.2). $\square$

The work criterion concerns the specific matrix H_ρ determined by the stationary Mellin state. Its nonexpansiveness is equivalent to the one-sided defect condition, and reflection then yields equality. Norm identities for other carrier maps apply through an explicit identification of their actions and metrics.

30. Finite prime-threshold values and reported checks

The first thresholds display the retained staircase directly. The values follow from $Q_q = \prod_{p < q} p$, $\delta_q = \varphi(Q_q)/Q_q$, and Theorem 14.2.

Threshold q	Q_q	$\varphi(Q_q)$	δ_q	$\text{Var } R_q$
2	1	1	1	2
3	2	1	1/2	2
5	6	2	1/3	4
7	30	8	4/15	16
11	210	48	8/35	96

For example, admitting 7 carries R_7 to R_{11} by

$$R_{11}(x) = R_7(x) - R_7(x/7), \quad Q_{11} = 7Q_7, \quad \delta_{11} = (1 - 1/7)\delta_7.$$

These are finite identities for the real argument x of the remainder readout. The prime admissions themselves are indexed by the ordered prime integers. The group-algebra induction and the Mellin handoff apply to every subsequent threshold.

30.1. Reported finite-identity checks

Table 1 reproduces the companion numerical report of 420 finite-identity checks. The group-algebra, remainder, and Mellin identities have the symbolic proofs given above.

The companion numerical audit reports 420 successful checks out of 420 across sixteen CSV outputs grouped as $T1 - T15$. T13 contributes 79 checks for the successive-prime, residual, and circular-readout identities. T14 contributes 34 checks for Section 16 and Corollary 17.8: the dilation action is evaluated on real-valued remainder functions over a grid containing noninteger x , the prime-admission operators commute and expand correctly, the finite

subset/product schema reproduces subsequent threshold remainders, and the Mellin orbit agrees to the stated high-precision tolerances. T15 contributes 53 checks for the structural audit. These computations check finite instances of the displayed group and induction identities.

The reported checks concern finite arithmetic transport and its scalar observations. The T14 implementation specifically verifies $S(p)R(q;\cdot)(x) = R(q;x/p)$ on the real carrier, rather than an integer-grid surrogate that would discard fractional arguments. The reported sup-norm diagnostic has tied first values, and the double-precision equilateral handoff uses a 3×10^{-14} tolerance. Neither alters an exact theorem identity. The finite identities are the subject of this audit. Stationary localization is governed by the work criterion in Proposition 29.11.

Table 1. Numerical audit of the finite identities. T8 uses a relative tolerance for one large-magnitude distance-ratio trajectory. T9 and T10 remain calibration/negative-control diagnostics only. T13 audits the prime-registry identities (79/79); T14 audits the real-variable prime-scaling/group-algebra schema and finite Mellin orbit of Section 16 and Corollary 17.8 (34/34); T15 audits the prime-registry identities (53/53). The combined suite reports 420/420 successful checks. Exact symbolic proofs in the text remain primary.

Test	Calibration	Checks	Passed
T1	Möbius/floor identity, periodicity and bound	5	5
T2	Mellin continuation: Dirichlet sum vs exact staircase integral	4	4
T3	Staircase Mellin convergence rate at the first zeta zero	6	6
T4	Reflection multiplier and $C = 1$ -display unit modulus	4	4
T5	Finite telescoping, zero tail identity, reciprocal scalar audit	13	13
T6	Frozen Gaussian/Gamma slot, theta duality, Hermite-2 check	7	7
T7	Auxiliary phase-window endpoints, lengths and successor phases	17	17
T8	Word-action distance/derivative cocycles and order dependence	11	11
T9	Landau registry-occupancy calibration	11	11
T10a	Davenport–Heilbronn reflected-pair/tail witness	8	8
T10b	Coefficient-cone diagnostic	24	24
T11	Structural and finite-registry identities	30	30
T12	Staircase, affine-relabeling and equilateral-rate identities	114	114
T13	successive-prime, residual, and circular-readout identities	79	79

Test	Calibration	Checks	Passed
T14	Real-variable prime-scaling/group schema and finite Mellin orbit	34	34
T15	prime-registry identities and singleton-readout audit	53	53

31. Consequences for the prime-word construction

Successive prime admission gives exact finite arithmetic evolution. Its normalized Mellin state is stationary exactly at a zeta zero and retains the exponent through $\rho = 1 + 1/K(\rho)$. The scalar value zero does not exhaust the information in the staircase readout.

Affine display changes preserve these statements when the function, kernel, and reflection are transported together. In the canonical variable the reflection line is $\Re s = 1/2$. Recoding changes its numerical label while preserving the localization claim.

The finite identities prove exhaustive prime admission, stationarity, residual propagation, and reflected pairing. Proposition 29.10 gives the exact equivalence between canonical localization and $\mathcal{D}(\rho) = 0$ at every nontrivial zero. The arithmetic registry contains every prime independently of the chosen analytic display.

Part III

Arithmetic bridge theorems

32. Goldbach as an exact finite survivor problem

A direct finite formulation treats Goldbach as an exact survivor problem and avoids importing a general non-collapse bridge.

Fix an even integer $N \geq 4$ and define the candidate set

$$\mathcal{C}_0(N) = \{a \in \mathbb{Z} : 2 \leq a \leq N - 2\}.$$

For a prime $p \leq \sqrt{N}$, retain a at the p -stage iff each of a and $N - a$ is either equal to p or is not divisible by p . Let $\mathcal{C}_p(N)$ denote the survivors after all primes up to p have been applied.

Theorem 32.1 (Finite Goldbach survivor identity). After sieving by every prime $p \leq \sqrt{N}$,

$$\mathcal{C}_{\leq \sqrt{N}}(N) = \{a : 2 \leq a \leq N - 2, a \text{ prime}, N - a \text{ prime}\}.$$

Hence the Goldbach representation count is exactly

$$G(N) = \frac{\#\mathcal{C}_{\leq \sqrt{N}}(N) + \mathbf{1}_{\{N/2 \text{ prime}\}}}{2}$$

when ordered pairs are identified under $a \leftrightarrow N - a$ and a central pair is counted once.

Proof. If a is composite and $2 \leq a \leq N - 2$, it has a prime divisor $p \leq \sqrt{a} \leq \sqrt{N}$, and since $a \neq p$ the p -stage removes it. The same holds for $N - a$. Thus every final survivor has both entries prime. Conversely a prime a is never removed: for a stage prime $p \neq a$, $p \nmid a$, and at $p = a$ the exceptional “equal to p ” clause preserves the prime itself. The same holds for $N - a$. Therefore the final survivors are exactly the prime pairs. The involution $a \mapsto N - a$ has two-element orbits except for $a = N/2$, which survives exactly when $N/2$ is prime. Counting these orbits gives the displayed correction. $\square$

Corollary 32.2 (Exact reduction of Goldbach). Goldbach’s conjecture is equivalent to the finite nonemptiness statement

$$\mathcal{C}_{\leq \sqrt{N}}(N) \neq \emptyset \quad \text{for every even } N \geq 4.$$

The vertices and eliminations are fixed by divisibility, including the root exceptions. The next theorem counts these survivors in the target interval and expresses nonemptiness as an exact one-sided discrepancy estimate.

Theorem 32.3 (Local sieve compatibility and exact interval accounting). For even $N \geq 4$ put

$$Q_N = \prod_{p \leq \sqrt{N}} p, \quad \mathcal{R}_N = \{r \in \mathbb{Z} : 0 \leq r < Q_N, p \nmid r(N-r) \text{ for all } p \mid Q_N\}.$$

Set $\nu_p(N) = 1$ if $p \mid N$, and 2 otherwise. Then

$$M_N := |\mathcal{R}_N| = \prod_{p \mid Q_N} (p - \nu_p(N)) > 0.$$

Define

$$A_N = \sum_{r \in \mathcal{R}_N} \left(\left\lfloor \frac{N-2-r}{Q_N} \right\rfloor - \left\lfloor \frac{1-r}{Q_N} \right\rfloor \right), \quad e_N = \#\{p \leq \sqrt{N} : p, N-p \text{ prime}\}.$$

The native survivor set satisfies

$$|\mathcal{C}_{\leq \sqrt{N}}(N)| = A_N + 2e_N - \mathbf{1}_{\{N=4\}}.$$

Thus full-period admissible residues are always nonempty, and their occurrence in the required interval is retained exactly by A_N .

Proof. Modulo p , the forbidden residues are 0 and N , coinciding exactly when $p \mid N$. The Chinese remainder theorem multiplies the allowed counts. At $p = 2$, evenness leaves one residue; at every odd prime at most two residues are excluded. Hence $M_N > 0$.

The floor difference counts integers congruent to r in $[2, N-2]$. An integer there with no prime divisor at most $\sqrt{N}$ is prime. Thus A_N counts exactly the pairs with neither entry a sieving prime. The remaining ordered pairs have a prime entry $p \leq \sqrt{N}$ and are counted by $2e_N$. Both entries can be at most $\sqrt{N}$ only if $N \leq 2\sqrt{N}$, hence only at $N = 4$. Its pair $(2, 2)$ requires the displayed correction. $\square$

The interval discrepancy is also exact:

$$A_N = \frac{(N-3)M_N}{Q_N} + \Delta_N, \quad \Delta_N = \sum_{r \in \mathcal{R}_N} \left(\left\{ \frac{1-r}{Q_N} \right\} - \left\{ \frac{N-2-r}{Q_N} \right\} \right), \quad |\Delta_N| < M_N.$$

The local factors are positive. Positivity in the target interval is exactly the one-sided inequality (68.1), which retains both the interval discrepancy and the exceptional prime roots.

33. Birch–Swinnerton-Dyer: exact bridge formulation

Let $E/\mathbb{Q}$ be an elliptic curve and put

$$r_{\text{an}} = \text{ord}_{s=1} L(E, s), \quad r_{\text{alg}} = \text{rank } E(\mathbb{Q}).$$

The arithmetic task is to construct a map from analytic multiplicity slots to rational generators, with independence and spanning proved for its image.

Definition 33.1 (Cancellation and generator spaces). Let C_E be the formal $\mathbb{Q}$ -vector space with one basis slot for each unit of vanishing order r_{an} , so $\dim_{\mathbb{Q}} C_E = r_{\text{an}}$. These are multiplicity slots, not independent tangent directions of a one-variable function. Let

$$G_E := E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

so $\dim_{\mathbb{Q}} G_E = r_{\text{alg}}$.

Theorem 33.2 (BSD bridge theorem). If there exists a canonical linear isomorphism

$$\mathfrak{B}_E : C_E \xrightarrow{\sim} G_E$$

compatible with the local Euler-factor registry and global closure, then

$$\text{ord}_{s=1} L(E, s) = \text{rank } E(\mathbb{Q}).$$

If, in addition, the globally obstructed locally admissible sectors form a finite set and this set is identified bijectively with the Tate–Shafarevich obstruction classes, then $\text{Sha}(E/\mathbb{Q})$ is finite.

Proof. An isomorphism preserves dimension, hence $r_{\text{an}} = r_{\text{alg}}$. The second assertion is immediate from the assumed bijection with a finite obstruction-sector set. $\square$

Remark 33.3 (The exact arithmetic construction required). To construct $\mathfrak{B}_E$, assign rational points to the analytic multiplicity slots and prove their independence and spanning in G_E . Finiteness of the identified obstruction classes gives the second conclusion. The full BSD formula also requires the leading coefficient, the period and regulator factors, and the local arithmetic factors.

34. Hodge: anchored graph representatives and Chow closure

Anchored graph representatives connect a closed analytic trace to its algebraic cycle and cohomology class.

Theorem 34.1 (Graph-space Hodge criterion). Let X be a smooth projective complex variety and

$$\eta \in H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X).$$

If η admits a non-destructive anchored graph representative whose anchored projection is a closed analytic trace $Y \subset X$ of pure codimension k and whose induced multiplicities represent η , then η is algebraic.

Proof. Because X is projective and Y is a closed analytic subvariety, Chow's theorem implies that Y is algebraic. With the multiplicities inherited from the anchored graph representative, Y defines an algebraic cycle Z_η . By the representative hypothesis, its cohomology class is $[Z_\eta] = \eta$. $\square$

Corollary 34.2 (Exact Hodge reduction). The Hodge conjecture follows if every rational (k, k) -class admits such an anchored graph representative.

The representative construction must produce closed analytic traces, their rational multiplicities, and the class identity $[Z_\eta] = \eta$. The following two constructions carry out this step for divisors and for products of projective spaces.

Proposition 34.3 (Divisor representatives on every smooth projective variety). Every rational $(1, 1)$ -class on a smooth projective complex variety has a rational divisor representative and a closed anchored graph representative. This is the classical codimension-one theorem.

Proof. Choose an integral lift c of a positive integer multiple $m\eta$. Its image in $H^2(X, \mathcal{O}_X) = H^{0,2}(X)$ vanishes. The exponential sequence supplies a holomorphic line bundle with first Chern class c ; projectivity makes it algebraic. A nonzero rational section gives a divisor $D = \sum_j n_j Y_j$ with class c [34]. Thus D/m represents η .

Take the graphs $\Gamma_j = \{(y, y) : y \in Y_j\} \subset X \times X$ with coefficients n_j/m . They are closed, and their first projections are isomorphisms onto Y_j . Their projected cycle has class η , including signed multiplicities. $\square$

The exponential sequence and a rational section thus supply the required trace and multiplicities in codimension one.

Proposition 34.4 (Constructed Hodge representatives on products of projective spaces). For $X = \prod_{i=1}^r \mathbb{P}_{\mathbb{C}}^{n_i}$, every rational class in $H^{2k}(X, \mathbb{Q})$ has an explicit rational algebraic-cycle representative; in particular every rational Hodge class does.

Proof. Complex projective n -space has one cell in each even real dimension $0, 2, \dots, 2n$. The hyperplane class h generates its cohomology ring $\mathbb{Q}[h]/(h^{n+1})$: transverse intersections

of j coordinate hyperplanes give h^j , and $\int_{\mathbb{P}^n} h^n = 1$. The product cell decomposition and Künneth formula therefore give

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[h_1, \dots, h_r] / (h_1^{n_1+1}, \dots, h_r^{n_r+1}).$$

Write the desired class uniquely as $\eta = \sum_{|a|=k} c_a \prod_i h_i^{a_i}$, with $0 \leq a_i \leq n_i$. Choose a coordinate linear subspace $L_{i,a_i} \cong \mathbb{P}^{n_i-a_i} \subset \mathbb{P}^{n_i}$. Then the rational cycle

$$Z_\eta = \sum_{|a|=k} c_a \left[\prod_i L_{i,a_i} \right]$$

has class η . These subvarieties are closed and algebraic, and rational coefficients allow signs. This finite sum constructs the representative for every class on the specified product. $\square$

Part IV

Geometry and field theory

35. Poincare: theorem and registry compatibility

Perelman's theorem supplies the topological input. A homeomorphism then transports the finite incidence and refinement structure.

Theorem 35.1 (Poincare). Every closed simply connected three-manifold is homeomorphic to S^3 .

Proof. This is Perelman's theorem obtained from Ricci flow with surgery and geometrization. $\square$

Proposition 35.2 (Registry transport of the established topology). Let M be a closed simply connected three-manifold equipped with any finite triangulation and any coherent finite refinement system on that triangulation. If $h : M \rightarrow S^3$ is a homeomorphism, then h transports the refinement incidence data to an isomorphic refinement system on S^3 .

Proof. Apply h to every cell, vertex, and incidence relation at each finite stage. Since a homeomorphism is bijective and preserves adjacency and closure, each descendant relation is carried to a descendant relation and every finite chain is transported bijectively. Thus the transported finite registry represents the same incidence and refinement structure. $\square$

The compatibility follows from the homeomorphism and holds at every finite refinement depth.

36. Yang–Mills: a rigorous compact transverse gap theorem

Let Ω be a compact connected transverse configuration domain and consider

$$D_T = -\alpha\Delta_\perp + U_T, \quad \alpha > 0,$$

where the boundary realization is specified by the closed quadratic form $\alpha \int |\nabla \psi|^2 + \int U_T |\psi|^2$ on a dense form domain $\mathcal{V} \subset H^1(\Omega)$ compactly embedded in $L^2(\Omega)$. Periodic, Neumann, and Dirichlet realizations on compact regular domains fit this formulation. The associated operator is self-adjoint with compact resolvent. Let $U_T \geq 0$ be bounded and positive on a set of positive measure. Define, with $\psi \in \mathcal{V}$,

$$\Delta_T = \inf_{\psi \neq 0} \frac{\alpha \int_\Omega |\nabla_\perp \psi|^2 + \int_\Omega U_T |\psi|^2}{\int_\Omega |\psi|^2}.$$

Theorem 36.1 (Positive compact-sector gap). Under the preceding hypotheses, if the only zero-gradient states are constants and U_T is positive on a set of positive measure, then $\Delta_T > 0$.

Proof. Because D_T is self-adjoint with compact resolvent, the infimum is attained by a normalized ground state ψ_0 . If $\Delta_T = 0$, both nonnegative terms in the numerator vanish. Hence $\nabla \psi_0 = 0$, so ψ_0 is constant, and also $U_T |\psi_0|^2 = 0$ almost everywhere. Since $U_T > 0$ on a set of positive measure, the constant must be zero, contradicting normalization. Therefore $\Delta_T > 0$. $\square$

Remark 36.2. The compact transverse operator has the positive gap Δ_T in its declared form domain. A field-theoretic extension requires construction of the physical state space and Hamiltonian, together with a comparison carrying this coercive estimate to that operator.

Proposition 36.3 (Explicit quantitative compact gap). Let Ω have volume V and the mean-zero Poincaré bound $\|w\|_2^2 \leq C_P \|\nabla w\|_2^2$. If $U_T \geq u_0 > 0$ on a measurable set E of measure $a > 0$, then

$$\Delta_T \geq \min \left\{ \frac{\alpha}{C_P(1 + 2V/a)}, \frac{au_0}{2V} \right\} > 0.$$

On the flat periodic torus of side L , the Poincaré constant is explicitly $C_P = L^2/(4\pi^2)$. On that torus, for constant $U_T = u_0$ the gap is exactly u_0 .

Proof. Write $\psi = m + w$ where $m = V^{-1} \int \psi$ and $\int w = 0$. On E , $|m|^2 \leq 2|\psi|^2 + 2|w|^2$, whence

$$\|\psi\|_2^2 = \|w\|_2^2 + V|m|^2 \leq C_P(1 + 2V/a) \|\nabla \psi\|_2^2 + \frac{2V}{au_0} \int U_T |\psi|^2.$$

Multiplication by the stated minimum bounds the right side by the energy numerator. For the flat torus, expand w in nonzero Fourier modes: every squared frequency is at least $(2\pi/L)^2$, proving the claimed Poincaré bound directly. On that torus, when $U_T = u_0$, the energy is at least $u_0 \|\psi\|_2^2$ and a constant test function attains equality. Finally, a nonnegative potential positive on a set of positive measure admits some such E, u_0 , because $\{U_T > 0\} = \bigcup_{n \geq 1} \{U_T \geq 1/n\}$. The explicit torus realization therefore removes the qualitative compactness and attainment requirements for this lower-bound proof. The estimate provides the explicit coercive constant for the declared compact-sector realization. $\square$

37. Hofer–Wysocki–Zehnder: orbit alternatives and registry transport

The cited contact-topological theorem gives the orbit alternatives. Its rigidity statement in the two-orbit case also determines the consequence of a degenerate orbit.

Theorem 37.1 (Two or infinity). Let Y be a closed connected three-manifold with a contact form λ such that $c_1(\ker \lambda)$ is torsion. The Reeb flow of λ has either two or infinitely many simple periodic orbits [35]. No nondegeneracy hypothesis is required.

Proof. This is the theorem of Cristofaro-Gardiner, Hryniewicz, Hutchings, and Liu [35], applied with the explicitly stated torsion first Chern class. $\square$

Corollary 37.2 (Degenerate-orbit consequence in the stated contact class). Under the preceding torsion first-Chern-class hypothesis, a degenerate periodic Reeb orbit, including a degenerate iterate, implies infinitely many simple periodic orbits.

Proof. Corollary 1.7 and its proof in [35] show that in the two-orbit case both simple orbits and all their iterates are irrationally elliptic, hence nondegenerate. A degenerate orbit excludes that case, and the dichotomy supplies the conclusion. This uses the cited rigidity theorem. $\square$

Proposition 37.3 (Registry compatibility). Let $\mathcal{O}$ be the set of simple Reeb orbits and encode each orbit as a closure state in a finite-support loop registry. Then the established two-or-infinity conclusion is preserved under any injective closure encoding $\Psi : \mathcal{O} \rightarrow S$: the image has cardinality two or infinite.

Proof. Injective maps preserve finite cardinality and infinitude. Hence $|\Psi(\mathcal{O})| = |\mathcal{O}|$. $\square$

Corollary 37.2 combines the contact rigidity theorem with the orbit dichotomy, and the injective encoding preserves the resulting alternatives.

Part V

Logic, computation, and measurement

38. Motion-complete observations and exact nonlinear quotients

A geometric carrier, an observation of that carrier, and an algorithm acting on its representation are different objects. The Chronoscalar terminology used here assigns *motion* to a finite succession of admissible events and *wake* to a future-relevant obligation retained after an observation has compressed the past. Neither a nonzero wake nor a nonlinear update alone is identified with a lower bound on unrestricted computation. The theorems below first determine the information an exact update requires. Section 39 then proves an unconditional exponential lower bound for a specified explicit-coordinate output contract, together with its constructive upper bound; the later closure results retain their own distinct interfaces and resource measures.

The finite-refinement, prime-word, and incompressible constructions give a common setting for the pointwise and certificate calculations below. Classical proof-complexity benchmarks are identified by attribution: the Tseitin width–size result uses Ben-Sasson–Wigderson [5]; the positive fractional-knapsack functional is Grigoriev’s construction [6], with harmonic organization as in [7,8]. Their stated operation models fix the resource measured by each benchmark.

38.1. Pointwise factorization without a linear carrier

Let S_m, S_{m+1} be state sets and let $F_m : S_m \rightarrow S_{m+1}$ be an event map. No addition, differentiability, or continuous interpolation is required. Let $\pi_m : S_m \rightarrow Q_m$ and $\pi_{m+1} : S_{m+1} \rightarrow Q_{m+1}$ be observations.

Theorem 38.1 (Nonlinear pointwise quotient criterion). There is a map $\overline{F}_m : \pi_m(S_m) \rightarrow Q_{m+1}$ satisfying

$$\overline{F}_m \pi_m = \pi_{m+1} F_m$$

if and only if

$$\pi_m(s) = \pi_m(t) \implies \pi_{m+1}(F_m(s)) = \pi_{m+1}(F_m(t))$$

for all $s, t \in S_m$. The map is unique on the observed image.

Proof. Necessity follows by applying the single-valued map $\overline{F}_m$ to equal observed inputs. For sufficiency, set $\overline{F}_m(\pi_m(s)) = \pi_{m+1}(F_m(s))$. The displayed fiber condition makes this

independent of the representative. It also fixes every value on the observed image, proving uniqueness. $\square$

A pair with equal present observations and different successor observations is therefore an exact obstruction to that representation, not a numerical approximation error. In a linear realization, the criterion becomes the kernel-transport condition used in the finite cochain model. It does not require an arbitrary deterministic algorithm to use that particular realization.

38.2. Closing an observation under finite motion

Let $\mathcal{A}$ be a finite alphabet of operations and let $F_a : S \rightarrow S$ for $a \in \mathcal{A}$. A partial transition system may be totalized with an invalid state and an invalid observation. Fix $h : S \rightarrow Y$. For a word $u = a_1 \cdots a_k$ put $F_u = F_{a_k} \cdots F_{a_1}$ and define

$$s \equiv_k t \iff h(F_u(s)) = h(F_u(t)) \text{ for all } |u| \leq k.$$

Then $\equiv_{k+1} \subseteq \equiv_k$.

Proposition 38.2 (Finite motion refinement).

$$s \equiv_{k+1} t \iff h(s) = h(t) \text{ and } F_a(s) \equiv_k F_a(t) \text{ for every } a \in \mathcal{A}.$$

Proof. Every nonempty word of length at most $k + 1$ begins with one label a and has at most k subsequent labels. The empty word gives the present observation. These are exactly the conditions displayed. $\square$

Write $s \equiv t$ when every finite continuation gives the same observation. The canonical map $q : S \rightarrow Q_h = S/\equiv$ has well-defined updates $[s] \mapsto [F_a(s)]$ and observation $[s] \mapsto h(s)$.

Theorem 38.3 (Minimal exact motion quotient). If an exact representation $\pi : S \rightarrow Z$ preserves h and all event updates, then there is a unique map $r : \pi(S) \rightarrow Q_h$ with

$$q = r\pi.$$

In particular $|Q_h| \leq |\pi(S)|$ when the sets are finite.

Proof. If $\pi(s) = \pi(t)$, exact successive updates give equal represented states after every finite continuation, and exact readout gives equal observations. Hence $s \equiv t$. Define $r(\pi(s)) = [s]$. The implication just proved gives well-definedness; surjectivity and uniqueness follow from the definition of q . $\square$

Theorem 38.4 (Terminating construction of complete motion memory). Let S be nonempty and finite, $n = |S|$, and let b_0 count the nonempty fibers of h . For the finite operation alphabet above, there is $k \leq n - b_0$ with $\equiv_k = \equiv_{k+1} = \equiv$. Thus the minimal exact motion quotient is constructible by finitely many refinement rounds without another closure hypothesis.

For a d -dimensional linear realization, observation D , and finite family of event matrices M_a , the complete observation is constructed by

$$V_0 = \text{row } D, \quad V_{k+1} = V_k + \sum_{a \in \mathcal{A}} V_k M_a.$$

This stabilizes by $k \leq d - \text{rank } D$. A row-basis matrix O of the stable space satisfies, for computable S, R_a ,

$$D = SO, \quad OM_a = R_a O \quad (a \in \mathcal{A}),$$

and retains every future observation.

Proof. Refine a partition by assigning each state the signature comprising its present block and the blocks of all immediate successors. The refinement proposition gives $\equiv_{k+1}$. Every strict refinement increases the block count, bounded by n . Once two consecutive partitions agree, the same rule gives agreement forever, so the stable partition is exactly equality of all future observations.

In the linear case, induction identifies V_k with the span of all rows DM_u for words of length at most k . Every strict increase raises dimension, bounded by d . The stable space is invariant under right multiplication by each M_a , giving the factorizations. Its kernel is the intersection of all future observation kernels. No future-relevant wake distinction is discarded. Row reduction constructs O, S, R_a . $\square$

These bounds use the size of the explicitly represented set or vector space; they are not polynomial bounds in a separate succinct input description. Arbitrary time-dependent sequences retain the finite-horizon construction below unless a fixed generating family or controller is included.

The factorization direction is $S \rightarrow \pi(S) \rightarrow Q_h$, not the reverse. This is the same future-behavior organizing principle as Myhill–Nerode, but the cardinal conclusion is relative to the specified continuation interface. It is not a statement about the running time of all algorithms. For a fixed finite horizon only the remaining permitted words are needed; quantification over all finite words does not require an infinitely executed process.

Motion determines the *observable quotient*, not necessarily the whole carrier. For example $F(x, z) = (f(x), g(z))$ with $h(x, z) = x$ never reveals z , even when g is nonlinear. Full recovery requires the additional condition that some allowed continuation distinguishes every pair of different states.

Theorem 38.5 (Constructed finite-horizon observation closure). Fix $N < \infty$, deterministic events $E_k : S_k \rightarrow S_{k+1}$ and arbitrary observations D_k . Put $E_{k:k} = I$ and $E_{j:k} = E_{j-1} \cdots E_k$ for $j > k$. The finite tuple

$$\hat{D}_k(x) = (D_j E_{j:k} x)_{j=k}^N$$

has a well-defined successor on its image: delete its first entry. It preserves the original observation and satisfies the nonlinear fiber criterion without an additional hypothesis. Any representation preserving these observations and all remaining events factors onto $\hat{D}_k(S_k)$.

If the state spaces are finite-dimensional vector spaces and the maps are linear, stack the matrices $D_j E_{j:k}$ into O_k . Then

$$\ker O_k = \bigcap_{j=k}^N \ker(D_j E_{j:k}), \quad E_k \ker O_k \subseteq \ker O_{k+1}.$$

Row reduction gives an equivalent observation with at most $\dim S_k$ coordinates.

Proof. The tuple for $E_k x$ consists exactly of entries $k+1, \dots, N$ of the tuple for x . Equal tuples therefore have equal successor tuples. The first coordinate is $D_k x$. Equal states in any other exact representation yield equal future observations by successive application of its event and readout maps; assigning their common tuple is consequently well defined and unique. The kernel formulas follow by setting every stacked coordinate to zero. A basis of the row space has the same kernel and recovers every original row by a fixed linear combination. $\square$

This construction changes an insufficient observation by retaining its needed future information. It does not turn a false factorization of the original D_k into a true one, and it supplies no cost-free computation of future entries. For example, $D(x, y) = x$ and $E(x, y) = (xy, 0)$ satisfy $E(\ker D) \subseteq \ker D$, but $(1, 0)$ and $(1, 1)$ have equal present observations and unequal successor observations. Thus the linear kernel test cannot replace the fiber test for nonlinear events.

38.3. Elliptic interaction, loss, and exact relative-state repair

Fix a nonsingular curve

$$E : y^2 = x^3 + Ax + B$$

over $\mathbb{Q}$. Its rational group law supplies exact finite arithmetic operations [9]. The Mordell–Weil decomposition organizes rational points after generators are known; it does not make rank computation or generator discovery free. In the calculations here, the input consists of the specified rational coordinates, and cost is measured in their bit lengths.

For $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$, the addition formulas yield

$$x(P + Q) - x(P - Q) = -\frac{4y_1y_2}{(x_2 - x_1)^2}, \quad (38.1)$$

$$x(P + Q) + x(P - Q) = \frac{2[(x_1 + x_2)(x_1x_2 + A) + 2B]}{(x_2 - x_1)^2}. \quad (38.2)$$

Indeed, subtract or add the expressions with squared numerators $(y_2 - y_1)^2$ and $(y_2 + y_1)^2$, and use the curve equations. The individual x -coordinates determine y_1^2, y_2^2 , but not generally the relative sign of y_1y_2 . Equation (38.1) identifies the lost coupling. Equation (38.2) shows that retaining $x(P - Q)$ repairs the generic addition update exactly.

For a concrete fiber failure, take $E : y^2 = x^3 - 2$, $P = (3, 5)$, observation $h(R) = x(R)$ with $h(O) = \infty$, and event $F(R) = R + P$. Then

$$h(P) = h(-P) = 3, \quad 2P = \left(\frac{129}{100}, -\frac{383}{1000}\right), \quad F(-P) = O.$$

Hence $h(F(P)) \neq h(F(-P))$. Theorem 38.1 forbids an x -only successor law for that event. The involution $R \mapsto -R$ preserves the present observation but not this observed successor.

Theorem 38.6 (Two translated slices recover an elliptic point). Fix $Q = (a, b) \in E(\mathbb{Q})$ with $b \neq 0$. The map

$$H_Q(P) = (h(P), h(P + Q))$$

is injective. If $P = (x, y)$ with $x \neq a$ and $z = x(P + Q)$, then

$$y = \frac{x^3 + Ax + B + b^2 - (x - a)^2(z + x + a)}{2b}.$$

Proof. The group law gives $(x - a)^2(z + x + a) = (y - b)^2$. Substitute $y^2 = x^3 + Ax + B$ and solve for y . The point O is identified by its first observation. The points with $x = a$ are Q and $-Q$; their second observations are $x(2Q)$ and ∞ , respectively. Since $b \neq 0$, $2Q \neq O$, so they are distinct as well. $\square$

For the translation orbit $P_m = P_0 + mQ$, put $x_m = h(P_m)$. Away from the exceptional coordinate cases, equation (38.2) gives the finite-memory law

$$x_{m+2} = \frac{2[(x_{m+1} + a)(ax_{m+1} + A) + 2B]}{(x_{m+1} - a)^2} - x_m. \quad (38.3)$$

Thus two successive observations close the nonlinear motion. Exceptional cases are handled by the exact group law. This proves information sufficiency, not a polynomial bound in orbit depth: a fixed number of coordinate slots does not bound the bit lengths of their contents. Section 39 treats a separate doubling orbit and proves that its prescribed explicit-coordinate realization has exponential output cost despite a fixed generative rule.

If all translations are allowed and the terminal observation is $[P = O]$, every two distinct points are future-distinguishable: translating both by $-P$ distinguishes P from $R \neq P$. Nevertheless one exact group law handles all transitions. Complete pointwise distinguishability therefore does not force an expanded table of separate computational rules.

38.4. Pointwise verification and existential closure are different observations

Let $V(x, w)$ be a polynomial-time verifier with witnesses padded to length $p(|x|)$. For a prefix u , its pointwise future function is $A_{x,u}(z) = V(x, uz)$ on completions of the required length.

Proposition 38.7 (Uniform pointwise generator). Every such verifier has an exact continuation representation of polynomial encoding length and polynomial pointwise evaluation cost: store (x, u) , append the next supplied bit, and evaluate $V(x, uz)$ on a supplied completion z .

Proof. The code has length at most $|x| + p(|x|) + O(1)$. Appending a bit is an exact update. The verifier supplies polynomial-time evaluation by definition. Equal codes cannot lose a future distinction because they retain the prefix itself. $\square$

The closure observation is instead

$$C_x(u) = \bigvee_{z \in \{0,1\}^{p(|x|)-|u|}} V(x, uz).$$

For a proper prefix,

$$C_x(u) = C_x(u0) \vee C_x(u1).$$

For SAT, polynomial-time computation of all such residual closure observations is equivalent to polynomial-time SAT decision: use the empty prefix in one direction, and restrict the input formula by u in the other. The standard verifier and decision distinction is described in [10].

A fixed equality verifier already demonstrates the cardinality limitation. For

$$G_m(X, Y) = \bigwedge_{i=1}^m (x_i \leftrightarrow y_i),$$

the 2^m prefixes $X = a$ have distinct continuation functions, since $Y = a$ accepts exactly the matching prefix. One class requires only m bits to name. Interleaving x_i, y_i gives a one-bit temporary comparison with an accumulated mismatch flag. More generally a depth- j witness cut has at most 2^j classes, whose names require at most j bits; this does not assert that canonical class names are efficiently computable.

Even every nonlinear interaction order can coexist with an inexpensive generator. For

$$F_m(z) = 1 - \prod_{i=1}^m (1 - z_i),$$

there are $2^m - 1$ nonzero expanded monomials and

$$\text{cr}_r F_m(e_{i_1}, \dots, e_{i_r}) = (-1)^{r+1} \neq 0$$

for distinct directions. This follows by summing $(-1)^{r-|S|}$ over nonempty subsets, since every corresponding Boolean input makes $F_m = 1$. The product expression still evaluates in $O(m)$ operations. The lower-bound task concerns existential closure of a succinct family, not generation of its pointwise table.

39. Explicit elliptic readout: lower and upper bounds

The two-slice theorem establishes information sufficiency for a nonlinear update. It does not bound the number of bits occupying those coordinate slots along a long orbit. This section specifies an explicit-output problem and proves both its unavoidable exponential readout cost and a construction within a logarithmic factor of that output-size lower bound. The event here is doubling, not the fixed translation used in equation (38.3). Every arithmetic statement concerns a finite event index.

39.1. The fixed binary-output interface

Fix once and for all

$$E : \quad y^2 = x^3 - 2, \quad P_0 = (3, 5), \quad P_j = [2^j]P_0.$$

The integers associated with the orbit will be shown below to satisfy

$$x(P_j) = \frac{a_j}{b_j}, \quad \gcd(a_j, b_j) = 1, \quad a_j, b_j > 0.$$

For a positive integer u , put

$$\text{bits}(u) := \lfloor \log_2 u \rfloor + 1,$$

and let $\text{bin}(u)$ denote its ordinary binary expansion without leading zeros.

Definition 39.1 (Explicit elliptic-readout function). For $m \geq 0$, define

$$\text{ECRead}(1^m) := \text{bin}(a_m) \# \text{bin}(b_m), \quad L_m := \text{bits}(a_m) + \text{bits}(b_m) + 1. \quad (39.1)$$

Here $\#$ is a separator. The unary input has length m ; the case $m = 0$ is the empty unary string. Computation is measured on a deterministic multitape bit-cost machine with a sequential append-only output tape and a fixed bounded emission rate. Work-tape representations are unrestricted, and every initialization, update, conversion, and output operation is charged.

The curve and seed are fixed data, not growing input parameters. The required output is the reduced numerator and denominator themselves. An orbit code, a rational expression, or a program producing the coordinates is not a replacement for that output string.

39.2. Exact doubling and the finite no-cancellation invariant

For $P = (x, y)$ with $y \neq 0$, the rational group law gives

$$\lambda = \frac{3x^2}{2y}, \quad x([2]P) = \lambda^2 - 2x.$$

Using $y^2 = x^3 - 2$ therefore yields the specialized doubling law [9]

$$x([2]P) = \frac{x^4 + 16x}{4(x^3 - 2)}. \quad (39.2)$$

Define the integer recurrence, before appealing to fraction reduction, by

$$\begin{aligned} a_0 &= 3, & b_0 &= 1, \\ a_{j+1} &= a_j^4 + 16a_jb_j^3, \end{aligned} \quad (39.3)$$

$$b_{j+1} = 4b_j(a_j^3 - 2b_j^3). \quad (39.4)$$

Lemma 39.2 (Finite positivity, coprimality, and orbit realization). For every $j \geq 0$, the recurrence satisfies

$$a_j > 0, \quad b_j > 0, \quad a_j^3 > 2b_j^3, \quad \gcd(a_j, b_j) = 1, \quad (39.5)$$

and

$$a_j \text{ is odd}, \quad 3 \mid a_j, \quad 3 \nmid b_j. \quad (39.6)$$

Moreover, P_j is a finite rational point with nonzero y -coordinate, and a_j/b_j is exactly its reduced x -coordinate. No greatest-common-divisor cancellation occurs in any recurrence step.

Proof. All integer assertions hold for $(a_0, b_0) = (3, 1)$, since $3^3 - 2 = 25 > 0$. Suppose they hold for (a, b) , and set

$$U = a^4 + 16ab^3 = a(a^3 + 16b^3), \quad V = 4b(a^3 - 2b^3).$$

The inductive inequality $a^3 > 2b^3$ gives $U, V > 0$. The integer U is odd and divisible by 3. Since $b^4 \equiv 1 \pmod{3}$,

$$V = 4b(a^3 - 2b^3) \equiv -8b^4 \equiv 1 \pmod{3}.$$

Thus V is not divisible by 3.

Suppose a prime p divides both U and V . If $p \mid b$, then $U \equiv a^4 \not\equiv 0 \pmod{p}$, using $\gcd(a, b) = 1$. Thus $p \nmid b$. Since U is odd, $p \neq 2$. Hence $p \mid V$ implies

$$a^3 - 2b^3 \equiv 0 \pmod{p}.$$

The prime cannot divide a , because that congruence would then imply $p \mid 2b^3$, contrary to $p \neq 2$ and $p \nmid b$. Therefore $p \mid U$ further implies

$$a^3 + 16b^3 \equiv 0 \pmod{p}.$$

Subtracting gives $18b^3 \equiv 0 \pmod{p}$, and hence $p \mid 18$. Only $p = 3$ remains. But $V \equiv 1 \pmod{3}$, so V is not divisible by 3. This contradiction establishes $\gcd(U, V) = 1$.

To propagate strict positivity of the curve denominator, put

$$W = a^6 - 40a^3b^3 - 32b^6.$$

The following exact polynomial identity is obtained by expansion:

$$\boxed{U^3 - 2V^3 = W^2.} \quad (39.7)$$

Equivalently, its left side is

$$a^3(a^3 + 16b^3)^3 - 128b^3(a^3 - 2b^3)^3.$$

Since $3 \mid a$ and $3 \nmid b$,

$$W \equiv b^6 \equiv 1 \pmod{3}.$$

Thus $W \neq 0$ and $U^3 > 2V^3$. This completes the simultaneous induction for all integer invariants.

Finally $P_0 = (3, 5)$ lies on E . If P_j has x -coordinate a_j/b_j , then $a_j^3 > 2b_j^3$ gives

$$y(P_j)^2 = (a_j/b_j)^3 - 2 > 0.$$

The doubling formula is therefore defined and produces a finite rational point with x -coordinate U/V . The coprimality proof shows that this is already reduced. Induction identifies the entire integer recurrence with the elliptic doubling orbit and excludes every exceptional doubling denominator on that orbit. $\square$

In particular, the invariant proves directly that

$$0 < \frac{b_j}{a_j} < 2^{-1/3} < 1. \quad (39.8)$$

In particular $a_j > b_j > 0$ at every event. The stronger invariant $a_j^3 > 2b_j^3$ gives the strict ratio bound needed below.

39.3. Sharp finite growth estimate and normalized accounting

Proposition 39.3 (Reduced-coordinate growth). At every event,

$$\boxed{a_j^4 < a_{j+1} < 9a_j^4.} \quad (39.9)$$

Writing $h_j = \log_2 a_j$, one has for all $j \geq 0$

$$4^j \log_2 3 \leq h_j \leq 4^j \log_2 3 + \frac{4^j - 1}{3} \log_2 9. \quad (39.10)$$

Both inequalities are strict when $j \geq 1$ and are equalities when $j = 0$. Consequently

$$\text{bits}(a_j) = \Theta(4^j), \quad L_j = \Theta(4^j). \quad (39.11)$$

Proof. Equation (39.3) and the ratio bound give

$$a_{j+1} = a_j^4 \left[1 + 16 \left(\frac{b_j}{a_j} \right)^3 \right], \quad 0 < 16 \left(\frac{b_j}{a_j} \right)^3 < 8.$$

This proves (39.9). Put

$$\varepsilon_j := \log_2 \left[1 + 16 \left(\frac{b_j}{a_j} \right)^3 \right].$$

Then

$$h_{j+1} = 4h_j + \varepsilon_j, \quad 0 < \varepsilon_j < \log_2 9. \quad (39.12)$$

Finite iteration yields

$$h_j = 4^j \log_2 3 + \sum_{r=0}^{j-1} 4^{j-1-r} \varepsilon_r.$$

The geometric sum is $(4^j - 1)/3$, proving (39.10), including the equality convention at $j = 0$.

Since $0 < b_j < a_j$,

$$h_j \leq L_j \leq 2h_j + 3.$$

Thus the numerator and the complete prescribed output have the lengths stated in (39.11). These are lengths of reduced integers, by Lemma 39.2, not lengths of avoidably unreduced formulas. $\square$

For the normalized quantity $\hat{h}_j = 4^{-j} h_j$, the same finite recurrence is

$$\hat{h}_{j+1} - \hat{h}_j = 4^{-j-1} \varepsilon_j, \quad \log_2 3 \leq \hat{h}_j \leq \log_2 3 + \frac{\log_2 9}{3}. \quad (39.13)$$

The normalized ledger is bounded, while $h_j = 4^j \hat{h}_j$ is exponentially large. Normalization accounts for the motion; it does not remove the bit length of the required explicit state. No limiting physical process is part of these finite identities.

39.4. Unconditional lower bound and a near-output-optimal construction

Theorem 39.4 (Explicit elliptic-readout complexity). For the function and machine interface of Definition 39.1, the output length is $L_m = \Theta(4^m)$. Every correct deterministic algorithm Π satisfies

$$T_\Pi(m) = \Omega(L_m) = \Omega(4^m). \quad (39.14)$$

There is one fixed uniform exact algorithm A with

$$T_A(m) = O(L_m \log(L_m + 2)) = O((m + 1)4^m). \quad (39.15)$$

The upper bound uses binary multiplication in $O(s \log(s + 2))$ bit operations on a multitape Turing machine [11]. The lower bound requires no multiplication theorem and no restriction on the algorithm's internal representation.

Proof. The output-length statement is Proposition 39.3. Let κ_Π bound the number of output symbols written in one step of the fixed machine implementing Π . Correctness requires emission of the entire string in (39.1), so

$$\kappa_\Pi T_\Pi(m) \geq L_m.$$

This proves the lower bound for every correct algorithm, including one that never explicitly follows the intermediate doubling states.

For the upper bound, initialize $(a, b) = (3, 1)$ and scan the unary input. For each input symbol, compute from the old values

$$u = a^4 + 16ab^3, \quad v = 4b(a^3 - 2b^3),$$

and replace (a, b) by (u, v) . Emit $\text{bin}(a) \# \text{bin}(b)$ after the scan. The invariant lemma proves correctness and guarantees that no greatest-common-divisor reduction is needed. A fixed collection of work tapes suffices for the powers, products, and sums, with all copying and tape traversal charged.

At stage j , the operands and the finitely many intermediate products have $O(4^j)$ bits. The update uses a fixed number of additions, subtractions, shifts, and products. With the multiplication routine of [11], its bit cost is

$$O(4^j \log(4^j + 2)) = O((j + 1)4^j).$$

Therefore

$$\sum_{j=0}^{m-1} O((j + 1)4^j) = O((m + 1)4^m).$$

Reading the unary input and emitting L_m output symbols are within this bound. Finally $L_m = \Theta(4^m)$ and $\log(L_m + 2) = \Theta(m + 1)$ convert it to the first form in (39.15). The constant-size case $m = 0$ is included by the displayed $m + 1$ and $L_m + 2$ conventions. $\square$

The bound is a logarithmic-factor window relative to output length: $\Omega(L_m)$ for every correct implementation and $O(L_m \log(L_m + 2))$ for the constructed one. The integer recurrence is implemented in the accompanying code; the asymptotic upper bound uses the cited multiplication theorem.

39.5. Output obligation, internal compression, and complete traces

Proposition 39.5 (Output-obligation telescope). In a correct computation of $\text{ECRead}(1^m)$, let e_t be the number of final-output symbols emitted by a stage boundary t and define

$$\Phi_t = L_m - e_t.$$

If an event segment costs c_t steps and the output rate is at most κ_Π symbols per step, then

$$\kappa_\Pi c_t + \Phi_{t+1} \geq \Phi_t. \quad (39.16)$$

Since $\Phi_0 = L_m$ and $\Phi_T = 0$ at successful termination, total cost is at least L_m/κ_Π .

Proof. The segment emits at most $\kappa_\Pi c_t$ symbols. Thus $e_{t+1} - e_t \leq \kappa_\Pi c_t$, which is precisely (39.16). Sum over segments; the intermediate potentials telescope. $\square$

This is specifically an *output-obligation* potential. It measures a necessary component of implemented work, not a universal measure of internal semantic difficulty. Here unit weights suffice: nonlinear arithmetic establishes the required length, and sequential emission converts that length into the unconditional lower bound.

Corollary 39.6 (Construction and required readout). Suppose a uniform encoder produces any exact internal description $q_m = \text{Enc}(1^m)$ and a decoder must output the string (39.1). Then

$$T_{\text{Enc}}(m) + T_{\text{Dec}}(q_m) = \Omega(4^m).$$

Proof. The composition is a correct algorithm for the same explicit-output function, so Theorem 39.4 applies. The internal code need not itself be long. $\square$

For comparison, fix a trace interface that emits the strings for stages $0, \dots, m$ in order, with one separator between successive strings. Its required length is

$$L_m^{\text{trace}} = \sum_{j=0}^m L_j + O(m) = \Theta(4^m).$$

The output-length lower bound remains $\Omega(4^m)$, and the same recurrence emits each pair when available with total cost $O((m+1)4^m)$. This trace contract does not impose intermediate execution on an algorithm solving only the endpoint contract.

39.6. Exact finite output lengths

Direct exact integer iteration gives the following finite values. The last column includes the single separator required by (39.1).

Event j	Numerator bits	Denominator bits	L_j
0	2	1	4
1	8	7	16
2	32	26	59
3	125	122	248
4	499	497	997
5	1994	1993	3988
6	7975	7973	15949

These values support implementation regression testing. Coprimality, positivity, asymptotic output growth, and both time bounds follow from the general proofs, not the table. The accompanying checks also expand (39.7) symbolically and compare initial recurrence values against rational group-law doubling.

Remark 39.7 (Internal representations and required output). The theorem concerns ECRead and its declared ordinary-binary output, not all exact descriptions of the orbit point. A succinct code such as (E, P_0, m) , or a projective, recursive, or symbolic internal representation, is permitted. It does not satisfy the output contract by itself. Every correct implementation must still emit the same $\Theta(4^m)$ reduced-coordinate bits and hence incur $\Omega(4^m)$ time; the explicit construction establishes the stated upper bound as well. No theorem that all internal representations are large is required.

The unconditional lower bound quantifies over every correct implementation of ECRead. The finite nonlinear generator has fixed description length and bounded normalized accounting, while the required reduced-coordinate output has $\Theta(4^m)$ bits. The output obligation alone converts that proved growth into the runtime lower bound.

40. Uniform operational cost and normalized lower transport

The relevant resource belongs to one fixed uniform program over all input lengths. Let $\mathfrak{M}$ be a deterministic bit-cost model. An operational realization specifies finite encodings, initialization, updates, and readout. Exactness means correctness for every input in the declared language; when the realization also claims to represent a chosen continuation interface, it must preserve that interface and intertwine its events. A general decider need not reproduce a particular witness-prefix quotient.

40.1. Exact extensions, quotients, and fixed programs

A quotient may merge only states with equal designated future behavior. An exact auxiliary-state extension may instead introduce useful information: on valid states, a projection $p : \widehat{S} \rightarrow S$ and lifted events satisfy

$$p\widehat{F}_a = F_ap.$$

The initialization of the extra coordinates and every update of them are charged. Finite workspace, relative coordinates, running parity, and shared arithmetic circuits are all permitted. A lower bound restricted to quotients but excluding ordinary auxiliary state would not cover general deterministic computation.

For a fixed scheme Π , let

$$C_\Pi(N) = \max_{|x|=N} \text{Time}_\Pi(x), \quad S_\Pi(N) = \max_{|x|=N} \text{Space}_\Pi(x).$$

Here time includes initialization, all representation maintenance, recomputation, and readout. Bit length of written constants and states is charged in this model. A fixed program's description is independent of N ; uniformity does not permit an independently chosen new program for each input length.

The pointwise envelope $\inf_\Pi C_\Pi(N)$ is only a numerical envelope. It need not be attained by one scheme for all N . In particular,

$$\forall N \exists \Pi_N \quad \text{does not imply} \quad \exists \Pi \forall N.$$

A program may hard-code a finite length and use a general decider elsewhere. Counting its description blocks free large tables, but does not by itself interchange the quantifiers. All uniform lower-bound statements below retain the fixed scheme outside the quantifiers over length.

Proposition 40.1 (Configuration realization). If $\mathfrak{M}$ includes encoded configurations and ordinary deterministic machine transitions with polynomial overhead, then every

time- $T(N)$ deterministic decider has an exact operational realization of cost $\text{poly}(N, T(N))$. Conversely, if every primitive of $\mathfrak{M}$ has a uniform bit-cost implementation and a complete realization has polynomial total implemented cost, its language belongs to P.

Proof. Encode the finite control, head positions, input interface, and tape cells visited so far. In T steps at most $O(N + T)$ cells are needed, and simulation plus final readout has polynomial cost. Conversely concatenate the implementations of initialization, updates, and readout. Their complete deterministic computation has the stated total cost. Neither direction identifies a machine with the minimal quotient of a separately prescribed witness tree. $\square$

This is model alignment, not a lower bound; it agrees with the standard machine formulation in [10]. The desired separation quantifiers are

$$\forall \Pi \text{ correctly deciding } L, \forall k \in \mathbb{N}, \exists^\infty N : C_\Pi(N) > N^k. \quad (40.1)$$

For a language in NP, this would imply $P \neq NP$. An explicit NP-complete closure language makes the target concrete but is not needed for the elementary final implication from $L \in NP \setminus P$.

40.2. A staged lower certificate without repeated future charges

An exact compressed state carries a residual decision obligation, not an unrepresented set-theoretic complement. A useful residual interface is $(I_m, \mathcal{U}_m, A_m)$, where $A_m(i, u)$ is the observation after a permitted continuation. Exact quotienting preserves it. For a closure problem, the obligation includes the existential observation, not merely evaluation on a supplied continuation.

Let a fixed strategy have stages $q_0, \dots, q_T$, costs $c_0, \dots, c_{T-1}$, and terminal readout cost c_T . The potential may depend on the represented configuration and its interface. Invariance under a recoding is not granted for free: the recoding is an event whose cost must satisfy the same inequality.

Theorem 40.2 (Pay-or-propagate lower ledger). Suppose $\Phi_j \geq 0, \delta_j \geq 0$, and every stage obeys

$$c_j + \Phi_{j+1} \geq \Phi_j + \delta_j, \quad c_T \geq \Phi_T.$$

Then

$$\text{Cost}(\Pi, x) := \sum_{j=0}^T c_j \geq \Phi_0 + \sum_{j < T} \delta_j.$$

Proof. Summation gives $\sum_{j<T} c_j + \Phi_T \geq \Phi_0 + \sum_{j<T} \delta_j$. Replace the terminal potential by the no-smaller terminal cost. Each future obligation appears once as an outgoing term and once as an incoming term; it is not charged anew at every stage. $\square$

For a fixed finite-horizon costed decision graph closed under admissible concatenation, the corresponding cost-to-go obeys

$$V_j(q) = \inf_{a \text{ admissible at } q} \{c(q, a) + V_{j+1}(F_a(q))\}.$$

This Bellman identity requires the state to include the information on which later admissibility depends. It does not authorize a nonuniform choice of a different free program at each input. A potential satisfying the preceding local inequalities is a lower certificate on that graph; defining it as the unknown optimal runtime merely restates the lower-bound task.

Theorem 40.3 (Normalized inverse-radial ledger). Let $g_j > 0$ for $0 \leq j \leq T$. Suppose

$$g_j c_j + \Psi_{j+1} - \Psi_j \geq g_j J_j \quad (j < T), \quad \Psi_T \leq g_T c_T.$$

Set $\Gamma = \max_{0 \leq j \leq T} g_j$. Then

$$\text{Cost}(\Pi, x) \geq \frac{\Psi_0 + \sum_{j<T} g_j J_j}{\Gamma}. \quad (40.2)$$

Proof. The weighted inequalities telescope to $\sum_{j<T} g_j c_j + \Psi_T \geq \Psi_0 + \sum_{j<T} g_j J_j$. The left side is at most $\Gamma \sum_{j \leq T} c_j$ by terminal domination. Divide by Γ . $\square$

If the weights arise from an adjoint radial or Sturm–Liouville recurrence, they still have to satisfy this normalization. Multiplying every weight by a large constant leaves the ordinary-cost lower bound unchanged. A superpolynomial weighted numerator is insufficient when the largest weight grows comparably. The computational analogue of the fluid telescope is therefore a normalized lower certificate, not reversal of an upper stability inequality.

40.3. Finite nonlinearity, the balanced filter, and its kernel

For maps between abelian groups define

$$\text{cr}_r F(v_1, \dots, v_r) = \sum_{S \subseteq [r]} (-1)^{r-|S|} F\left(\sum_{i \in S} v_i\right).$$

Affine maps have vanishing cross-effects of order at least two. Vanishing binary cross-effect means additivity up to a constant in the group setting; real linearity additionally requires the appropriate regularity or scalar compatibility. No converse from one selected probe to global affineness is assumed.

For equal-radius 120° vectors in a Euclidean plane, put

$$\Delta_{120}F(x) = \sum_{i=1}^3 F(x + v_i) - 3F(x), \quad \sum_i v_i = 0, \quad \sum_i v_i v_i^T = \frac{3r^2}{2}I.$$

Proposition 40.4 (Balanced finite filter and regularity-correct comparison). Every affine F satisfies $\Delta_{120}F = 0$. For a C^2 map near x ,

$$\Delta_{120}F(x) = \frac{3r^2}{4}\Delta F(x) + o(r^2)$$

componentwise. If its Hessian is locally Lipschitz, the remainder is $O(r^3)$.

Proof. The affine contribution is $A \sum_i v_i = 0$. The exact Taylor integral is

$$F(x + v) = F(x) + DF_x v + \int_0^1 (1-t) D^2 F_{x+tv} [v, v] dt.$$

Subtract the Hessian value at x inside the integral and sum the three terms. Contraction with the frame identity gives $3r^2 \Delta F(x)/4$. If $\omega(r)$ is a modulus of continuity of the Hessian, the residual norm is at most $3r^2 \omega(r)/2$, which is $o(r^2)$. A Lipschitz Hessian with constant L improves the bound to $Lr^3/2$. $\square$

The finite filter is primary; the smooth estimate is only a comparison. It has a nontrivial nonlinear kernel: $F(x_1, x_2) = x_1^2 - x_2^2$ satisfies $\Delta_{120}F = 0$ identically, because the two diagonal frame sums agree. The filter therefore isolates one aggregate nonlinear channel, not every nonlinearity. The value of this specified filter determines whether the occupied event contributes to the triadic channel.

40.4. Anticonformal separation and backward observation transport

A real plane may carry the metric-compatible quarter-turn

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J^2 = -I,$$

and the orientation-reversing involution

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C^2 = I, \quad CJ = -JC.$$

The quarter-turn J is not itself anticonformal. A scaled metric identity $T^T G' T = |\lambda|^2 G$ specifies a scaled isometry; its orientation or an additional relation $TJ = -J'T$ distinguishes anticonformal from conformal transport. None of these geometric identities

alone implies acceptance separation. That requires a continuation whose decision outcomes differ on the relevant partners.

The pointwise obstruction of Theorem 38.1 is the appropriate nonlinear semantic test. An expression such as $P'TJQ = 0$ belongs to an additional linear realization and must be proved for that realization; it cannot be inferred from the notation $P + Q = I$.

Backward accounting need not linearize the state dynamics. For an observable f and event F_a , the pullback $U_a f = f \circ F_a$ is linear in f , even when F_a is nonlinear in the state. Thus linear dual accounting can accompany nonlinear finite motion. A finite-dimensional, inexpensive, motion-invariant family of such observables is additional structure, not guaranteed by the pullback identity.

41. Finite nonlinear Boolean closure and exact affine absorption

The operations in this section are finite arithmetic over $\mathbb{F}_2$. This is a specified Boolean realization of nonlinear events, not an identification of finite-field addition with a physical fluid variable. The input length is the binary size of the explicit system or circuit, including indices and constants.

41.1. Exact gate lift and global affine elimination

Introduce a variable for every wire of an acyclic Boolean circuit. The gate equations are

$$\begin{aligned} z = u \oplus v &\iff z + u + v = 0, \\ z = \neg u &\iff z + u + 1 = 0, \\ z = u \wedge v &\iff z + uv = 0. \end{aligned}$$

XOR, AND, and constants express every Boolean circuit, since $u \vee v = u + v + uv$. The output condition is $z_{\text{out}} = 1$.

Theorem 41.1 (Unique finite event lift). Every assignment of circuit input variables extends uniquely to all gate variables satisfying the gate equations. The circuit has an accepting input if and only if the gate equations with $z_{\text{out}} = 1$ have a solution.

Proof. Process gates in topological order. At every gate, all input values are already determined and its equation fixes its output uniquely. Each displayed equation is exactly the gate truth table. The output condition therefore preserves acceptance in both directions. $\square$

This is the standard exact auxiliary-variable construction, not expansion of the entire computation tree [12]. Collect the affine equations, including any affine output requirement,

as $Ly = b$, and the multiplication equations as $q_i(y) = 0$. If $Ly = b$ is inconsistent, the complete system is inconsistent. Otherwise compute a particular solution y_0 and a basis matrix K for $\ker L$.

Theorem 41.2 (Exact residual reduction). Define $R_i(\theta) = q_i(y_0 + K\theta)$, where $\theta \in \mathbb{F}_2^d$ and $d = \dim \ker L$. Then

$$Ly = b, q_i(y) = 0 \text{ for all } i \iff R_i(\theta) = 0 \text{ for all } i$$

under the bijection $y = y_0 + K\theta$. Every residual equation has degree at most two after reducing $\theta_j^2 = \theta_j$.

Proof. The affine parametrization is bijective onto the compatible affine space. Substitution therefore preserves the remaining solution set exactly. Affine substitution into a quadratic polynomial does not increase degree, and Boolean reduction does not increase it either. $\square$

This global elimination absorbs the explicitly present affine constraints. It does not claim that nonlinear deductions cannot reveal additional affine consequences later.

41.2. Finite polar form and common absorption directions

Write each reduced residual as

$$R_i(\theta) = c_i + \sum_j a_{ij}\theta_j + \sum_{j < k} b_{i,jk}\theta_j\theta_k.$$

Define its finite polar form

$$\beta_i(u, v) = R_i(u + v) + R_i(u) + R_i(v) + R_i(0).$$

Direct expansion gives

$$\beta_i(u, v) = \sum_{j < k} b_{i,jk}(u_jv_k + u_kv_j).$$

It is bilinear and alternating. Moreover $\beta_i = 0$ identically if and only if R_i is affine, since $\beta_i(e_j, e_k) = b_{i,jk}$. Under an invertible affine recoding $\theta = A\xi + c$, the new form is $\tilde{\beta}_i(u, v) = \beta_i(Au, Av)$.

Theorem 41.3 (Cosetwise affine absorption). Let $H \subseteq \mathbb{F}_2^d$ satisfy $\beta_i(h, h') = 0$ for all $h, h' \in H$ and every i . Then every R_i is affine on every coset $a + H$. Given a basis for H of codimension r , solvability can be decided using 2^r linear-system solves and polynomial overhead.

Proof. For a basis $h_1, \dots, h_s$ of H , quadratic expansion gives

$$\begin{aligned} R_i \left(a + \sum_j \lambda_j h_j \right) &= R_i(a) + \sum_j \lambda_j (R_i(a + h_j) + R_i(a)) \\ &\quad + \sum_{j < k} \lambda_j \lambda_k \beta_i(h_j, h_k). \end{aligned}$$

The last line vanishes. There are 2^r cosets and the resulting constraints are linear in λ . Gaussian elimination tests each one. Bilinearity means a supplied basis for H can itself be checked on its basis pairs in polynomial time. $\square$

Finding H is a charged task. The total bound is

$$T(N) \leq T_{\text{find } H}(N) + 2^r \text{poly}(N).$$

For a gate residual $R_i = \ell_i m_i + a_i$ with affine factors, choose one factor ℓ_i from each product. The common kernel of their linear parts fixes all selected factors on each coset. Its codimension is their coefficient rank r , bounded by the number of multiplication gates. Enumerating these factor values therefore gives an exact $2^r \text{poly}(N)$ algorithm. Shared factors can make r much smaller than the number of expanded interaction terms.

41.3. Sharp affine-cover noncompression, and its nonlinear escape

Let

$$\Gamma_m = \{(a, b, c) \in \mathbb{F}_2^{3m} : c_i = a_i b_i \text{ for } 1 \leq i \leq m\}.$$

An exact affine cover is a union of affine subspaces, each contained in Γ_m .

Theorem 41.4 (Independent multiplication requires exactly 2^m affine pieces). The minimum exact affine-cover size of Γ_m is 2^m .

Proof. The one-gate graph consists of $(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1)$. Its three nonzero points are linearly independent, so these four points do not form an affine plane. Every affine subspace contained in this graph has dimension at most one.

If $A \subseteq \Gamma_m$ is affine, its image under projection to each gate triple has dimension at most one. The combined projections are injective, so the direction space of A has dimension at most m . Thus $|A| \leq 2^m$. Since $|\Gamma_m| = 2^{2m}$, at least 2^m pieces are necessary.

For each fixed a , the set $(a, b, \text{diag}(a)b)$ as b varies is an m -dimensional affine subspace. The 2^m choices of a partition Γ_m , attaining the bound. $\square$

Membership in Γ_m is nevertheless checked with m multiplications, and the output $c = \gamma$ has the explicit solution $a = \gamma, b = (1, \dots, 1)$. The affine-cover lower bound is therefore

not a deterministic-time lower bound. It proves a real failure of affine representation while exhibiting a compact nonlinear repair.

41.4. The exact nonlinear closure ledger

For the residual system of Theorem 41.2, use the Boolean-function algebra

$$\mathcal{B}_d = \mathbb{F}_2[\theta_1, \dots, \theta_d] / (\theta_1^2 - \theta_1, \dots, \theta_d^2 - \theta_d).$$

Every element has a unique multilinear representative and is determined by its values on the Boolean cube. To see injectivity, write $f = g + \theta_d h$ and evaluate at $\theta_d = 0, 1$, then induct on d .

Set

$$Z_0 = 1, \quad Z_j = \prod_{i=1}^j (1 + R_i), \quad D_j = Z_{j-1} R_j.$$

Theorem 41.5 (Finite constraint admission and global closure). For every stage,

$$Z_{j-1} = Z_j + D_j, \quad D_i D_j = 0 \ (i \neq j), \quad 1 = Z_t + \sum_{j=1}^t D_j.$$

Furthermore,

$$\exists \theta \ \forall i \ R_i(\theta) = 0 \quad \Longleftrightarrow \quad Z_t \neq 0 \text{ in } \mathcal{B}_d.$$

Proof. $Z_j(\theta) = 1$ exactly when the first j constraints vanish. The update follows by expanding $Z_j = Z_{j-1}(1 + R_j)$. If $i < j$, $D_i D_j$ contains $R_i(1 + R_i) = 0$, because every Boolean function is idempotent. Summation gives the partition identity. Finally a nonzero Boolean function is one on at least one Boolean assignment, which proves the closure criterion. $\square$

The circuit describing all prefixes Z_j and eliminations D_j has size $O(s + t)$ when the residual equations have total shared circuit size s . Each prefix is generated once and reused. Thus writing the nonlinear ledger is inexpensive; recognizing whether its final succinctly represented function vanishes on the whole cube is the closure task. A formal polynomial identity test without the Boolean relations is a different task.

For example $R_1 = 1$ yields $Z_1 = 0$ immediately and eliminates the entire cube. The number of assignments eliminated cannot itself be charged as computational work.

41.5. A complete nonabsorption law in the black-box model

Theorem 41.6 (Exact black-box existential closure). An exact deterministic value-query algorithm deciding whether an unknown function $b : \{0, 1\}^m \rightarrow \{0, 1\}$ has any value one needs 2^m queries in the worst case.

Proof. Run the algorithm on the all-zero function. If it leaves a location u unqueried, the function with a single one at u gives the same transcript but a different closure answer. Exactness therefore requires every location to be queried. Exhaustive querying attains the bound. $\square$

Along the all-zero transcript, let Φ_j be the number of unqueried locations. A new query decreases it by at most one; a recoding or repeated query does not decrease it. Charging one per query gives $c_j + \Phi_{j+1} \geq \Phi_j$, while exact zero termination requires $\Phi_T = 0$. Theorem 40.2 yields 2^m . This is a complete pay-or-propagate theorem in its stated access model. It is exponential in address length but linear in the size of an explicitly supplied truth table. An explicit SAT formula supplies structural information not available through those value queries, so the theorem is not transferred to SAT without an additional model simulation.

42. Explicit-model nonabsorption and global compatibility

The following benchmarks use explicit constraints, not merely query access. Each lower bound is stated with its operation class. Each inexpensive escape is a constructive test of that operation class, rather than an assumed impossibility of proving a stronger theorem.

42.1. Tseitin boundary retention under resolution

Let $G = (V, E)$ be a finite connected simple graph with $|V| = n$, bounded maximum degree d , and incidence matrix B over $\mathbb{F}_2$. Put a Boolean variable on each edge and prescribe charges b_v of odd total parity. The equations are

$$\bigoplus_{e \ni v} x_e = b_v.$$

Their sum is $0 = 1$, since each edge occurs twice. A degree- d_v parity equation has a CNF encoding with 2^{d_v-1} clauses of width d_v . This is the classical Tseitin construction [5].

For $S \subseteq V$, cancellation of internal edges gives the exact finite boundary identity

$$\bigoplus_{e \in \partial S} x_e = \bigoplus_{v \in S} b_v.$$

For a clause C , define $\mu(C)$ as the minimum number of vertex equations that imply it. Every proper vertex subsystem is satisfiable: a dependence among a proper subset of incidence rows would be a nonempty proper U with $\partial U = \emptyset$, contradicting connectedness. Hence $\mu(\perp) = n$, while each initial clause has $\mu \leq 1$.

Resolution combines $A \vee x$ and $D \vee \neg x$ into $A \vee D$. Supporting sets can be united, so a resolvent satisfies $\mu(C) \leq \mu(C_1) + \mu(C_2)$. Weakening cannot increase this minimum burden.

Lemma 42.1 (Pointwise boundary-retention witness). If S is a minimum-cardinality set of vertex equations implying a non-tautological clause C , then every edge in ∂S occurs in C .

Proof. Let a boundary edge e have endpoint $v \in S$. Minimality supplies an assignment satisfying $S \setminus \{v\}$ and falsifying C . It must violate the equation at v . If x_e were absent from C , flip that single bit. Only the equation at v changes within S , so all of S becomes satisfied while C remains false. This contradicts implication. $\square$

Suppose $|\partial S| \geq h \min(|S|, n - |S|)$ for a fixed $h > 0$. In any resolution refutation, take the first clause with $\mu \geq n/3$. Its parents each have burden below $n/3$, so it has burden below $2n/3$. The preceding lemma gives width at least $hn/3$.

The Ben-Sasson–Wigderson size–width theorem states, for M variables, initial width w_0 , minimum refutation width W , and resolution length S , that

$$\log S = \Omega \left(\frac{(W - w_0)^2}{M} \right)$$

in the nontrivial range [5]. With bounded degree and fixed expansion, $M = O(n)$, $w_0 = O(1)$ and $W = \Omega(n)$, giving $S = \exp(\Omega(n))$. This final conversion invokes the cited theorem; it is not inferred from width alone. The result applies to proofs required to use resolution, not all decision procedures.

The exact global escape is summing parity equations. Even total charge is sufficient for consistency: choose a spanning tree, set non-tree edges to zero, and process leaves toward the root, assigning the last incident edge to satisfy each leaf. The root equation is the total parity condition. This gives a linear number of graph operations and polynomial bit cost. The large resolution boundary and the small parity certificate coexist because they belong to different reasoning algebras.

42.2. A dual certificate for equality reasoning

Let Boolean constraints be $F_i(x) = 0$ over $\mathbb{Q}$, including $x_j^2 - x_j = 0$. A static equality certificate is

$$1 = \sum_i H_i(x) F_i(x).$$

Its degree is $\max_i \deg(H_i F_i)$ before cancellation. Let $\mathcal{V}_D$ be multilinear Boolean polynomials of degree at most D , and let $\mathcal{W}_D$ be the span of Boolean reductions of monomial multiples mF_i with $\deg m + \deg F_i \leq D$.

Lemma 42.2 (Finite dual nonabsorption). If $\Lambda : \mathcal{V}_D \rightarrow \mathbb{Q}$ satisfies $\Lambda(1) = 1$ and $\Lambda(\mathcal{W}_D) = 0$, no degree- D equality certificate exists. Conversely, if $1 \notin \mathcal{W}_D$, such a functional exists.

Proof. Apply Λ to a purported certificate to obtain $1 = 0$. Conversely extend a basis of $\mathcal{W}_D$ by 1, assign zero to the old basis and one to 1, and extend linearly. $\square$

This lemma is finite-dimensional separation, not an assertion that the functional is a true probability distribution or a machine-time lower bound.

42.3. Exactly-one matching: a sharp degree obstruction

Take $h + 1$ objects and h locations, $h \geq 2$, with variables x_{ij} . Define

$$E_i = \sum_{j=1}^h x_{ij} - 1, \quad C_{ikj} = x_{ij}x_{kj} \ (i < k), \quad B_{ij} = x_{ij}^2 - x_{ij}.$$

This is an exactly-one polynomial encoding of the pigeonhole contradiction. It is important to specify the encoding rather than transfer a degree bound from a differently encoded formula.

For a multilinear monomial x_S of degree $k \leq h$, define

$$\Lambda_h(x_S) = \begin{cases} 1/(h)_k, & S \text{ is a partial matching,} \\ 0, & S \text{ contains a repeated object or location,} \end{cases}$$

where $(h)_k = h(h-1) \cdots (h-k+1)$ and $(h)_0 = 1$.

Theorem 42.3 (Sharp equality-certificate degree for the specified matching encoding). The minimum static equality-certificate degree for $E_i = C_{ikj} = B_{ij} = 0$ is $h + 1$.

Proof. For a matching S of size $k \leq h - 1$, if object i is already assigned, exactly one term of $x_S \sum_j x_{ij}$ reduces to x_S and all other terms are incompatible. If i is unassigned, exactly $h - k$ locations remain, so

$$\Lambda_h \left(x_S \sum_j x_{ij} \right) = \frac{h-k}{(h)_{k+1}} = \frac{1}{(h)_k}.$$

For incompatible S , all terms have value zero. Thus $\Lambda_h(x_S E_i) = 0$ within the degree budget. Collision multiples have incompatible support, and Boolean multiples reduce to zero. Since $\Lambda_h(1) = 1$, Lemma 42.2 excludes degree at most h .

For the upper bound put $s_i = \sum_j x_{ij}$ and $P = \prod_{i=1}^{h+1} s_i$. Every expanded monomial of P assigns $h + 1$ objects to h locations, hence is divisible by a collision factor. Assign each monomial to one collision to obtain $P = \sum A_{ikj} C_{ikj}$. The product telescope gives

$$P - 1 = \sum_{i=1}^{h+1} (s_i - 1) \prod_{\ell < i} s_\ell.$$

Subtracting proves a certificate for 1, each summand having degree at most $h + 1$. $\square$

The upper bound is a degree construction; it does not assert that this expanded certificate is short.

42.4. The precise positivity failure and its cheap repair

Put $t_j = \sum_i x_{ij}$. The exact identity

$$-1 = \sum_{j=1}^h (1 - t_j)^2 - \sum_{i,j} B_{ij} - 2 \sum_j \sum_{i < k} C_{ikj} + \sum_i E_i \quad (42.1)$$

has degree two and polynomial description size. To verify it, expand the squares as $h - \sum_{i,j} x_{ij} + \sum B_{ij} + 2 \sum C_{ikj}$ and use $\sum_i E_i = \sum_{i,j} x_{ij} - (h + 1)$.

At a solution, equation (42.1) would equate -1 to a sum of nonnegative real squares. The equality-only functional fails this rule exactly:

$$\Lambda_h(t_j) = \Lambda_h(t_j^2) = \frac{h+1}{h}, \quad \Lambda_h((1 - t_j)^2) = -\frac{1}{h}.$$

Thus a valid global nonlinear operation defeats this particular dual certificate. The next section repairs positivity rather than merely increasing the functional's initial value.

43. Positive finite certificates for fractional closure

This section derives the central half-integer specialization of Grigoriev's fractional-knapsack bound [6]. The functional is classical. The finite conditioning identities, harmonic positivity calculation, and matching degree upper bound are written out to make the operational limitations inspectable. Harmonic multilinear organization on Boolean slices is developed in [7]; the distinction between truncated consistent moments and actual global distributions is also emphasized in [8].

Let $n = 2m + 1 \geq 3$, $r = n/2 = m + 1/2$, and $s = x_1 + \cdots + x_n$. The constraints are

$$x_i^2 - x_i = 0 \quad (1 \leq i \leq n), \quad s - r = 0.$$

They are inconsistent because s is integral on the Boolean cube. A *static equality-plus-sum-of-squares certificate* has the form

$$-1 = \sum_{\ell} p_{\ell}(x)^2 + a(x)(s - r) + \sum_i b_i(x)(x_i^2 - x_i). \quad (43.1)$$

Its degree is the maximum degree of the displayed summands before cancellation. This is not the degree of a dynamic polynomial-calculus derivation, which is a different proof measure.

43.1. Moment assignment and exact handoff

For a multilinear monomial x_S , define

$$\mathcal{L}_{n,r}(x_S) = \frac{(r)_{|S|}}{(n)_{|S|}}, \quad (a)_k = a(a-1) \cdots (a-k+1), \quad (a)_0 = 1, \quad (43.2)$$

and extend linearly after Boolean reduction. It is a real linear functional with rational values on rational polynomials; $\mathcal{L}_{n,r}(1) = 1$.

Lemma 43.1 (Equality annihilation). For $\deg q \leq n-1$,

$$\mathcal{L}_{n,r}((s-r)q) = 0.$$

All multiples of Boolean equations also have value zero after reduction.

Proof. For x_S with $|S| = k < n$, Boolean reduction gives $sx_S = kx_S + \sum_{i \notin S} x_{S \cup \{i\}}$. Put $a_k = (r)_k / (n)_k$. Since $(n-k)a_{k+1} = (r-k)a_k$,

$$\mathcal{L}_{n,r}((s-r)x_S) = (k-r)a_k + (n-k)a_{k+1} = 0.$$

Extend linearly; reduction never increases degree. □

For q in the remaining $n-1$ variables, the finite identities

$$\mathcal{L}_{n,r}(x_1 q) = \frac{r}{n} \mathcal{L}_{n-1,r-1}(q), \quad (43.3)$$

$$\mathcal{L}_{n,r}((1-x_1)q) = \frac{n-r}{n} \mathcal{L}_{n-1,r}(q) \quad (43.4)$$

follow by taking one factor out of the falling factorials. Hence

$$\mathcal{L}_{n,r}(f) = \frac{r}{n} \mathcal{L}_{n-1,r-1}(f|_{x_1=1}) + \frac{n-r}{n} \mathcal{L}_{n-1,r}(f|_{x_1=0}). \quad (43.5)$$

This is an exact finite certificate handoff. It is not a genuine probability process on all complete histories: sufficiently long conditioning can force a formal remaining target outside the allowable Boolean range. The positivity needed for a degree lower bound must be proved independently.

43.2. Finite harmonic pairing

Let P_k be the coefficient space of homogeneous multilinear polynomials of degree k , with inner product $\langle h, g \rangle_c = \sum_{|S|=k} h_S g_S$. A homogeneous polynomial $h = \sum_{|S|=k} h_S x_S$ is harmonic when

$$\sum_{S \supset T, |S|=k} h_S = 0 \quad \text{for every } |T| = k - 1.$$

Repeated summation gives the same identity for every $|T| < k$.

Lemma 43.2 (Harmonic pairing). For homogeneous harmonic h, g of degrees $k, \ell \leq m$,

$$\mathcal{L}_{n,r}(hg) = 0 \quad (k \neq \ell),$$

and, when $k = \ell$,

$$\mathcal{L}_{n,r}(hg) = \frac{(r)_k (n-r)_k}{(n)_{2k}} \langle h, g \rangle_c. \quad (43.6)$$

Proof. For monomials x_S, x_T , the pairing depends only on $|S \cap T|$, because $|S \cup T| = |S| + |T| - |S \cap T|$. Expand a function of this finite intersection count in the basis $\binom{|S \cap T|}{j}$.

Then

$$\sum_S h_S \binom{|S \cap T|}{j} = \sum_{\substack{U \subseteq T \\ |U|=j}} \sum_{S \supseteq U} h_S.$$

For $j < k$ this is zero. If $|T| < k$, every term vanishes, proving orthogonality. If $|T| = k$, only $j = k$ can survive and it contributes a scalar multiple of h_T . Thus the equal-degree pairing is a scalar multiple of the coefficient inner product.

Use $h_0 = \prod_{i=1}^k (x_{2i-1} - x_{2i})$ to determine the scalar. Its coefficient norm squared is 2^k . For integer $t \in \{0, \dots, n\}$, $\mathcal{L}_{n,t}$ is uniform averaging on assignments with exactly t ones. The value h_0^2 is one precisely when every displayed pair has one one and one zero. Counting gives

$$\mathcal{L}_{n,t}(h_0^2) = 2^k \frac{(t)_k (n-t)_k}{(n)_{2k}}.$$

Both sides are polynomials in t of degree at most $2k$. Since $n+1 > 2k$, agreement at $0, \dots, n$ proves the identity for arbitrary r . Division by 2^k gives (43.6). $\square$

For a self-contained decomposition, define raising and lowering operators on monomials by

$$Ux_S = \sum_{i \notin S} x_{S \cup \{i\}}, \quad Dx_S = \sum_{i \in S} x_{S \setminus \{i\}}.$$

They are adjoint in the coefficient inner product and on P_k satisfy

$$DU - UD = (n - 2k)I.$$

The off-diagonal terms cancel by pairing an added index with a removed one; the diagonal counts are $n - k$ and k . Therefore

$$\|Up\|_c^2 = \|Dp\|_c^2 + (n - 2k)\|p\|_c^2.$$

For $k < n/2$, U is injective. Orthogonal decomposition gives $P_j = \ker D \oplus U(P_{j-1})$ in the needed range. Iterating and using $Uh = (s - k)h$ for homogeneous degree k yields

$$p = \sum_{k=0}^m \sum_{\alpha} h_{k,\alpha} q_{k,\alpha}(s), \quad \deg q_{k,\alpha} \leq m - k, \quad (43.7)$$

for every $\deg p \leq m$, where the $h_{k,\alpha}$ form coefficient-orthonormal harmonic bases. This is a finite coefficient-space decomposition; it invokes no continuum motion.

43.3. Positivity on every permitted square

Theorem 43.3 (Truncated positivity). For $r = n/2$ and $\deg p \leq m$,

$$\mathcal{L}_{n,r}(p^2) = \sum_{k=0}^m \frac{(r)_k (n-r)_k}{(n)_{2k}} \sum_{\alpha} q_{k,\alpha}(r)^2 \geq 0. \quad (43.8)$$

Proof. In (43.7), replace each $q_{k,\alpha}(s)$ by $q_{k,\alpha}(r)$ to obtain p_r . The difference $p - p_r$ is divisible by $s - r$ with multiplier degree at most $m - 1$. Hence $p^2 - p_r^2$ is such a multiple with multiplier degree at most $2m - 1 < n$. Lemma 43.1 gives $\mathcal{L}(p^2) = \mathcal{L}(p_r^2)$. Apply Lemma 43.2. Every displayed prefactor is positive for $k \leq m$ because $r = n - r = m + 1/2$. This proves the formula and nonnegativity for every linear combination, not merely individual monomials. $\square$

Theorem 43.4 (Sharp static certificate degree). For $n = 2m + 1$ and $r = n/2$, the minimum degree of a static certificate (43.1) is exactly $n + 1$.

Proof. A certificate of degree at most n has each $\deg p_\ell \leq m$. Applying $\mathcal{L}_{n,r}$ gives -1 on the left, a nonnegative sum on the right, and zero for the constraint multiples. This is impossible.

For attainment, let $P_n(z) = \prod_{j=0}^n (z - j)$. On every Boolean assignment, s is an integer between 0 and n , so $P_n(s)$ vanishes. Boolean reduction therefore writes

$$P_n(s) = \sum_i b_i(x) (x_i^2 - x_i)$$

with each summand of degree at most $n + 1$. Put $Q_n(z) = (P_n(z) - P_n(r))/(z - r)$. Since $P_n(r) \neq 0$,

$$-1 = \frac{(s - r)Q_n(s)}{P_n(r)} - \sum_i \frac{b_i(x)}{P_n(r)} (x_i^2 - x_i). \quad (43.9)$$

This is a degree- $n + 1$ equality certificate and hence a certificate of form (43.1). $\square$

Theorem 43.4 is a static degree statement. Grigoriev's paper distinguishes static Positivstellensatz degree from dynamic calculus degree; no equality between those different measures is inferred here [6].

43.4. Stability under bounded-degree exact extensions

Theorem 43.5 (Polynomial-extension stability). Adjoin auxiliary variables $y_j = f_j(x)$, where the fully substituted polynomials have degree at most $d \geq 1$, and include their defining equations. The y_j may be real-valued; if Boolean axioms for them are added, assume f_j is Boolean-valued on the original Boolean cube. There is no extended static certificate of total degree D when $dD \leq n$. Consequently every such certificate satisfies $dD \geq n + 1$.

Proof. Substitute $y = f(x)$. Every defining equation vanishes identically; any additional Boolean auxiliary equation vanishes after Boolean reduction. A degree- D summand acquires degree at most dD . A square root of degree at most $\lfloor D/2 \rfloor$ acquires degree at most $d\lfloor D/2 \rfloor \leq m$. It therefore has nonnegative functional value by Theorem 43.3. The original constraint multiples remain within the annihilation budget. Applying $\mathcal{L}_{n,r}$ again contradicts $-1 \geq 0$. $\square$

For recursive extensions, d is measured after substituting the entire chain back to the original variables. It is not the maximum degree of one local defining equation. The distinction between base proofs and extended systems is substantive [13].

Proposition 43.6 (Full positivity cannot preserve an inconsistent finite system). There is no functional on the full Boolean-function algebra with $\mathcal{L}(1) = 1$, positivity on every square, and $\mathcal{L}((s - r)q) = 0$ for every Boolean function q .

Proof. For each assignment a , let e_a be its point indicator. Then $e_a^2 = e_a$ and $\sum_a e_a = 1$, so $w_a = \mathcal{L}(e_a) \geq 0$ and $\sum_a w_a = 1$. But $(s - r)e_a = (|a| - r)e_a$, giving $(|a| - r)w_a = 0$. The coefficient is never zero for half-integer r , so every $w_a = 0$, a contradiction. $\square$

The persistent certificate is therefore resource-bounded, not a genuine distribution on nonexistent solutions. The full decision remains easy: $2s = n$ has an even left side and odd right side on Boolean assignments. The next section makes the inexpensive algebraic generation of a high-degree certificate explicit as well.

44. Succinct nonlinear certificate generation and exact checking

The degree- $n + 1$ upper certificate in Theorem 43.4 was obtained by Boolean reduction. An expanded reduction may conceal its operational cost. We now construct the certificate by a

shared recurrence and give a deterministic checker using only low-dimensional polynomial identities. This is an additional derivation for the displayed family, not an assertion that arbitrary circuit identities admit the same checker. Circuit-based proof complexity distinguishes these representation questions explicitly [14].

44.1. A local nonlinear interpolation identity

For $k \geq 0$ define

$$P_k(u) = \prod_{j=0}^k (u - j).$$

For $k \geq 1$ set

$$H_k(u, z) = \frac{P_k(u + z) - (1 - z)P_k(u) - zP_k(u + 1)}{z^2 - z}. \quad (44.1)$$

Lemma 44.1 (Polynomial local residual). H_k is a polynomial in $\mathbb{Z}[u, z]$ of total degree at most $k - 1$. The exact identity

$$P_k(u + z) = (u - k + (k + 1)z)P_{k-1}(u) + (z^2 - z)H_k(u, z) \quad (44.2)$$

holds over $\mathbb{Z}[u, z]$.

Proof. The numerator in (44.1) vanishes at $z = 0$ and $z = 1$, so it is divisible by the monic polynomial $z(z - 1)$ over $\mathbb{Z}[u]$. Its total degree is at most $k + 1$: the possible degree- $k + 2$ terms in $z(P_k(u + 1) - P_k(u))$ cancel, since $P_k(u + 1) - P_k(u)$ has degree k in u . Division by $z(z - 1)$ gives total degree at most $k - 1$.

Now $P_k(u) = (u - k)P_{k-1}(u)$ and $P_k(u + 1) = (u + 1)P_{k-1}(u)$. Therefore

$$(1 - z)P_k(u) + zP_k(u + 1) = (u - k + (k + 1)z)P_{k-1}(u).$$

Substitute this into (44.1). □

For example, $H_1 = 1$ and $H_2(u, z) = 3u + z - 2$. The general construction uses ordinary exact polynomial division in only two variables; it does not expand polynomials in n original Boolean coordinates.

44.2. Transport the local residual through a prefix chain

Put $S_k = x_1 + \cdots + x_k$, with $S_0 = 0$, and define

$$A_k = S_{k-1} - k + (k + 1)x_k, \quad E_k = P_k(S_k).$$

Then $E_0 = P_0(0) = 0$ and Lemma 44.1 gives

$$E_k = A_k E_{k-1} + (x_k^2 - x_k) H_k(S_{k-1}, x_k). \quad (44.3)$$

This is an exact nonlinear handoff: each event supplies its Boolean defect, while the preceding defect is transported by the next affine factor.

Theorem 44.2 (Shared-generator Boolean-ideal identity). For every $n \geq 1$,

$$P_n(S_n) = \sum_{i=1}^n (x_i^2 - x_i) \beta_i(x), \quad \beta_i(x) = H_i(S_{i-1}, x_i) \prod_{k=i+1}^n A_k. \quad (44.4)$$

Each $\deg \beta_i \leq n - 1$. All coefficients β_i have a shared arithmetic-circuit representation of size polynomial in n .

Proof. Iterate (44.3) from $E_0 = 0$. The source inserted at stage i receives exactly the later transport product $\prod_{k=i+1}^n A_k$. This proves the identity. The degree bound is $(i - 1) + (n - i) = n - 1$.

Build all prefix sums S_k once, all affine factors A_k once, and all suffix products $\prod_{k=i+1}^n A_k$ once. The polynomial H_i has degree at most $i - 1$ in two variables and therefore $O(i^2)$ possible monomials. Generate its coefficient table exactly from (44.1), and evaluate that table at (S_{i-1}, x_i) by shared arithmetic gates. Summing these polynomial-size constructions over $i \leq n$ is polynomial. No expansion of the suffix products in the original variables is required. $\square$

44.3. Generate and check the sharp certificate in polynomial bit cost

Let n be odd and $r = n/2$. Use the β_i of Theorem 44.2 in (43.9), namely

$$-1 = \frac{(S_n - r)Q_n(S_n)}{P_n(r)} - \sum_{i=1}^n \frac{\beta_i(x)}{P_n(r)} (x_i^2 - x_i), \quad (44.5)$$

where $Q_n(u) = (P_n(u) - P_n(r))/(u - r)$.

Theorem 44.3 (Polynomial generation and structured verification). For the explicit half-integer family, certificate (44.5) has degree $n + 1$, a polynomial-size shared-circuit description, polynomial-size rational coefficients, and a deterministic polynomial-time generation and checking procedure.

Proof. The algebraic identity and degree follow from Theorems 43.4 and 44.2. Generate the univariate P_k , the bivariate H_k , the univariate Q_n , and the prefix/suffix circuit gates. Dense coefficient operations in a fixed one- or two-variable space of degree $O(n)$ take polynomially many arithmetic operations.

The integer coefficients of the products P_k and their substitutions in two variables have polynomial bit length: bounding the sum of coefficient magnitudes by products of $O(n)$ factors of size $O(n)$ gives an $O(n \log n)$ logarithmic bound. Monic division by $z^2 - z$ uses only coefficient additions and retains polynomial bit length. The values at $r = n/2$ and division by the nonzero rational $P_n(r)$ also have polynomial bit length, since they are products and divided differences of $O(n)$ rationals of polynomial bit length. Thus exact binary arithmetic has polynomial total cost.

The checker first verifies that the input is the specified Boolean half-integer system. It verifies $P_k(u) = (u - k)P_{k-1}(u)$, the bivariate identities (44.2), and $(u - r)Q_n(u) = P_n(u) - P_n(r)$ by coefficient comparison. It checks that the supplied circuit gates follow the prescribed prefix sums, factors, and suffix products of (44.4). These local checks imply the global identity by induction. They require no general polynomial-identity test on arbitrary multivariate circuits and no expansion of the final n -variable certificate. $\square$

The result is an exact operational separation:

minimum static degree $n + 1$ but polynomial generation and verification cost.

The nonlinearity in the recurrence is not denied. It is carried by a compact shared generator. The construction is specific to this family and does not supply a polynomial algorithm for general SAT.

44.4. Multiplicative depth is the degree-distortion ledger

Proposition 44.4 (Degree growth along a division-free circuit). If a division-free arithmetic circuit starts with variables of degree one and scalar constants, and its maximum multiplication depth is ℓ , then every output has formal total degree at most 2^ℓ .

Proof. An addition gate has degree at most the larger input degree; a multiplication gate has degree at most the sum of the two input degrees. Induct on gates, counting one additional multiplication level at a product. Cancellation can only lower actual degree. $\square$

Corollary 44.5 (Degree–extension-depth tradeoff). If all auxiliary coordinates in Theorem 43.5 are generated by a division-free circuit of multiplication depth at most ℓ , then every static extended certificate of degree D satisfies

$$2^\ell D \geq n + 1. \quad (44.6)$$

Proof. After full substitution, each auxiliary coordinate has degree at most 2^ℓ . Theorem 43.5 excludes $2^\ell D \leq n$. $\square$

This is a genuine lower tradeoff, but for fixed D it gives only a logarithmic depth requirement. A balanced product of n variables has degree n , depth $\lceil \log_2 n \rceil$, and $O(n)$ gates. The tradeoff therefore cannot be converted into a superpolynomial runtime bound by replacing degree with computational cost.

Likewise scalar amplitude and output cost must be separated. Repeated squaring starting from 2 generates 2^{2^k} with a circuit of k multiplications. Its expanded binary output has $2^k + 1$ bits and requires that much writing time when that output is demanded. A one-bit decision about the represented quantity need not demand that expansion. A resource theorem must state its required output interface.

44.5. Dual potentials with explicit operation normalization

Let $P_0 = 0$, $P_{j+1} = P_j + v_j$, and $P_T = 1$ be a particular certificate-construction process. Let a linear functional satisfy $\Lambda(1) = B > 0$. If every permitted increment satisfies $|\Lambda(v_j)| \leq c_j$, then

$$\Phi_j = |B - \Lambda(P_j)|$$

obeys $c_j + \Phi_{j+1} \geq \Phi_j$. Indeed the triangle inequality applied to $B - \Lambda(P_j) = B - \Lambda(P_{j+1}) + \Lambda(v_j)$ proves it. Since $\Phi_0 = B$ and $\Phi_T = 0$, total cost is at least B .

If the proved local bound is instead $|\Lambda(v_j)| \leq Kc_j$, the conclusion is $\sum c_j \geq B/K$. Rescaling Λ rescales both B and K , not the resulting cost lower bound. A coefficient-norm theorem may justify such a bound in a coefficient-charging proof model; it does not establish the bound for bit-cost computation with shared arithmetic circuits. Theorem 44.3 is a concrete test case any claimed universal operation bound must pass.

45. Certificate compilation and uniform decision cost

A certificate can be large in one measure, compact in another, easy to verify when supplied, and difficult to discover uniformly. The distinction between verification and generation must remain visible when a proof-complexity result is used as a candidate computational lower bound. Circuit descriptions and polynomial identity checking are separate ingredients in strong algebraic proof systems [14]; the structured checker of Theorem 44.3 handles its particular family without assuming a general circuit-identity algorithm.

45.1. Sound refutations and the generation problem

Let $\text{Check}(F, \pi)$ be a fixed deterministic relation decidable in time polynomial in $|F| + |\pi|$. It is a sound complete refutation system for CNF formulas if satisfiable F have no accepted

proof and every unsatisfiable F has some accepted proof. Define

$$S_{\text{Check}}(N) = \max_{\substack{|F| \leq N \\ F \text{ unsatisfiable}}} \min\{|\pi| : \text{Check}(F, \pi) = 1\}.$$

The maximum is over unsatisfiable inputs only; a value for an empty finite set may be set to zero.

Theorem 45.1 (Proof length versus efficient proof generation). The following statements concern different tasks.

A polynomial bound on S_{Check} gives a polynomial-size witness for unsatisfiability, not by itself a polynomial-time procedure to find that witness. Existence of some polynomially bounded sound complete refutation system is equivalent to $\text{NP} = \text{coNP}$.

If there exists a total deterministic polynomial-time generator G which, on every unsatisfiable F , returns a proof accepted by a fixed sound polynomial-time checker, then $\text{SAT} \in \text{P}$. Conversely, $\text{SAT} \in \text{P}$ implies the existence of such a generator and checker.

Proof. A polynomial proof-length bound makes unsatisfiability an NP language by guessing and checking the proof. Completeness of UNSAT then gives $\text{coNP} \subseteq \text{NP}$, and complementation gives equality. Conversely, $\text{NP} = \text{coNP}$ supplies an NP verifier for UNSAT, which is such a polynomially bounded refutation system. This is the Cook–Reckhow characterization, also stated in [5].

For the generation statement, run $G(F)$ and then check its output. If F is unsatisfiable the check accepts; if it is satisfiable soundness forbids acceptance. The total procedure decides SAT in polynomial time. Conversely, given a polynomial-time SAT decider, define a checker accepting an empty proof precisely when that decider says unsatisfiable. The empty-proof generator is total and polynomial-time. The checker is sound and complete because the assumed decider is correct. $\square$

The converse is an existence argument under $\text{SAT} \in \text{P}$, not an algorithm obtained here. A generator promised to halt only on unsatisfiable inputs is insufficient for the stated decision reduction unless a suitable total polynomial time bound is supplied.

45.2. Exact operation traces

For a fixed correct deterministic decider M , a rejecting computation trace consists of its successive configurations. A checker verifies the initial configuration, each local transition, and final rejection. Soundness as a refutation certificate uses the mathematical correctness of the fixed machine M ; the trace checker does not decide that global correctness from an arbitrary supplied machine code. The trace length is polynomial in the input length and runtime, for example $O(T(N)(N + T(N)))$ with a direct configuration encoding.

Thus an inexpensive decision gives an inexpensive trace in a proof system adapted to that correct decision rule. To transfer a lower bound from one selected proof system to every deterministic algorithm, a compiler into that selected system must be proved, not presumed.

Theorem 45.2 (Costed refutation-compiler transfer). Fix a sound complete refutation system Check. Suppose that for every correct deterministic SAT decider M there are a uniform compiler and constants a_M, K_M such that every rejecting run on F , of duration $T_M(F)$, produces an accepted proof of length at most

$$K_M(|F| + T_M(F))^{a_M}.$$

If $S_{\text{Check}}(N)$ exceeds every fixed polynomial for infinitely many N , then $\text{SAT} \notin \text{P}$.

Proof. If a correct M ran in polynomial time, the compiler would supply polynomial-length refutations of every unsatisfiable formula. The exponent and constant depend on that fixed M , not the input. This contradicts the assumed superpolynomial minimum-length bound for the fixed checker. $\square$

A uniform compiler for the chosen checker must also cover polynomial parity reasoning and the shared certificate generator of Theorem 44.3. Its construction must charge the translation of those operations into the target proof format.

45.3. Constructive certificate comparison

For the half-integer family, the following facts now coexist with complete proofs:

$$D_{\text{static SOS}} = n + 1, \quad 2^\ell D_{\text{extended}} \geq n + 1,$$

while the structured high-degree certificate has polynomial generation and verification cost. The conclusion is not that the positive functional was defective. It was valid throughout its degree budget. The conclusion is that degree is an operation-relative obstruction, and a shared generator can carry a high-degree certificate without paying for its full expansion.

This is an operational comparison with the two-slice elliptic example. A supplied relative coordinate repairs a pointwise update; a supplied proof repairs uncertainty about closure; and a uniformly constructible shared proof generator makes that repair algorithmic. Information sufficiency and total realization cost are separate. The explicit elliptic-output theorem in Section 39 proves an exponential cost when the required coordinate strings themselves grow exponentially; it is not weakened by the existence of a short orbit code.

46. Normalized cost certificates for decision closure

Theorem 39.4 establishes unconditional lower and upper bounds for explicit elliptic output. This section treats decision interfaces through a normalized certificate theorem. The finite ledger $Z_j = Z_{j-1}(1 + R_j)$ of Theorem 41.5 specifies the Boolean closure relation, and the positive functional controls finite polynomial consequences within its resource budget.

46.1. A calibrated lower certificate

Let Π be one fixed exact uniform algorithm deciding a chosen closure language. Let q_j^Π be its represented configurations, and charge actual bit operations. A candidate lower certificate consists of $\Psi_j^\Pi \geq 0$, $J_j^\Pi \geq 0$, and positive weights g_j^Π . It must be constructed without defining its value to be the unknown optimal runtime or supplying an uncharged closure oracle.

The local requirement is

$$g_j^\Pi c_j^\Pi + \Psi_{j+1}^\Pi - \Psi_j^\Pi \geq g_j^\Pi J_j^\Pi, \quad \Psi_T^\Pi \leq g_T^\Pi c_T^\Pi. \quad (46.1)$$

The functions may be model-relative and may depend on the represented state. An exact change of representation is itself a charged event. Arbitrary recodings need not preserve a numeric norm, and nonlinearity alone does not supply the operation bound.

Definition 46.1 (Uniform decision-cost certificate). For an explicitly encoded NP closure language, a uniform decision-cost certificate assigns data satisfying (46.1) to every exact uniform deterministic decider, including its nonlinear state extensions, shared algebraic generators, recomputation, and global transforms. Let

$$\Gamma_\Pi(x) = \max_{0 \leq j \leq T} g_j^\Pi.$$

The certificate is superpolynomial when, with the algorithm fixed before the input-length quantifiers,

$$\forall \Pi \forall k \in \mathbb{N} \exists^\infty N \exists x (|x| = N) : \frac{\Psi_0^\Pi + \sum_{j < T} g_j^\Pi J_j^\Pi}{\Gamma_\Pi(x)} > N^k. \quad (46.2)$$

Theorem 46.2 (Normalized decision-cost transfer). A superpolynomial uniform decision-cost certificate as in Definition 46.1, in a model polynomially equivalent to deterministic bit-cost computation, places its closure language outside P. If that language belongs to NP, then $P \neq NP$.

Proof. Theorem 40.3 bounds the actual cost below by the normalized expression in (46.2). Equation (40.1) follows for every correct decider. A polynomial-time decider would

contradict that statement for its fixed exponent and sufficiently large constant-absorbing power. If $L \in \text{NP} \setminus \text{P}$, the classes are unequal. $\square$

The transfer theorem uses the local operation bound and the quantified normalized demand as its two certificate properties. The explicit elliptic-output theorem has the concrete certificate $\Phi_t = L_m - e_t$ from Proposition 39.5, giving its unconditional lower bound directly.

46.2. Roles of pressure compatibility, anticonformal structure, and inverse-radial weights

In the finite-support incompressible construction, a Poisson/Hodge solve enforces the momentum constraint, and the retained wake has a specified event law. Its Wallis–Gregory theorem is an upper estimate for the stated recurrent class. The computational counterpart must permit every globally coordinated exact operation in its model, not only linear or locally chosen projections. Gaussian elimination, parity summation, and the shared certificate recurrence are concrete compatibility operations that a proposed obstruction must allow.

An anticonformal or branch-exchange realization can organize a semantic witness only after one proves that the exchanged states differ under the required closure observation. Even then, the cost of retaining the distinction remains separate. The two-slice elliptic result is a complete information repair, not a total-cost claim in orbit depth; the fixed doubling-output theorem proves expensive exact readout with a nearly matching construction. The independent multiplication graph is a complete exponential affine-cover obstruction with inexpensive nonlinear evaluation. Neither a 120° angle nor a nonzero cross-effect supplies a physical dissipation law or a deterministic-time charge.

Inverse-radial weights may transport a terminal lower certificate backward, but Theorem 40.3 governs their conversion to ordinary cost. The weight normalization and terminal condition cannot be dropped. The meaningful common architecture is therefore

finite event and observation $\longrightarrow$ exact compatibility or closure relation
 $\longrightarrow$ proved local resource inequality
 $\longrightarrow$ normalized global accounting.

46.3. Finite arithmetic examples

The accompanying exact-arithmetic scripts test finite instances of elliptic reconstruction, Boolean closure, affine absorption, the matching identity, fractional moments, and the local certificate recurrence. They also generate and verify a structured certificate. The separate elliptic-readout checks verify the polynomial positivity identity, reduced-coordinate invariants, initial group-law agreement, the binary-output contract, and finite growth bounds. These regression checks support implementation reproducibility; they do not replace the general proofs above.

Construction	Proved obstruction	Exact repair or scope
Elliptic x -observation	Translation need not descend to x alone	Two slices give information sufficiency, not a total depth-cost bound.
Explicit elliptic readout	$L_m = \Theta(4^m)$; every correct algorithm needs $\Omega(L_m)$ time	One fixed construction takes $O(L_m \log(L_m + 2))$ time.
Black-box closure	2^m worst-case value queries	Only a query-access theorem.
Independent multiplication	Exactly 2^m affine-cover pieces	$O(m)$ nonlinear evaluation.
Tseitin expander CNF	Exponential resolution refutation size	Global parity consistency is polynomial.
Exactly-one matching	Equality-certificate degree $h + 1$	Degree-two SOS identity.
Half-integer Boolean sum	Static equality-plus-SOS degree $n + 1$	Polynomial structured certificate generation.
Polynomial extensions	$dD \geq n + 1$, hence $2^\ell D \geq n + 1$	Degree distortion is not total bit cost.

47. Cantor refinement: exact non-collapse model

Theorem 47.1 (Cantor support). For the middle-third Cantor construction, level n has 2^n surviving cylinders of diameter 3^{-n} . The limit set is compact, uncountable, totally disconnected, and

$$\dim_H C = \frac{\log 2}{\log 3}.$$

Proof. Compactness follows from nested nonempty compact sets. The dimension follows from the regular refinement theorem with $b = 2$ and $\rho = 1/3$. Uncountability follows by identifying each branch with a binary sequence, and total disconnectedness follows from the persistent gaps created at every stage. $\square$

This is the canonical exact example of refinement without recovery of a primitive interval carrier. The embedding $C \subset [0, 1]$ is one-dimensional, while the refinement history retains a distinct branching structure.

48. Godel: exact logical theorem and registry translation

Theorem 48.1 (Godel–Rosser incompleteness). Every consistent effectively axiomatized theory containing Robinson arithmetic is incomplete.

Proof. Proof codes include finite certificates for axiom-enumeration steps, so proof checking is primitive recursive and numeralwise representable. The diagonal lemma supplies R saying that every proof of R has a no-larger coded proof of $\neg R$.

If R has proof code p , consistency excludes proofs of $\neg R$. Arithmetic verifies p and the finite list of failed opposite proof codes at most p . It proves $\neg R$, contradicting consistency.

If $\neg R$ has proof code q , consistency excludes proofs of R . Arithmetic verifies the finite list of failed R -proof codes below q . For any code at least q , the verified opposite proof q supplies the witness required by R . These two ranges prove R , again a contradiction. Thus neither sentence is provable. This is Rosser’s refinement of the standard diagonal argument. $\square$

Proposition 48.2 (Encoding invariance of incompleteness). Let Ψ be an injective map from sentences of such a theory T into a finite-support registry, and suppose registry closure is defined to mean theoremhood in T . Then an undecidable sentence remains non-closing in the registry.

Proof. If φ is undecidable, neither φ nor $\neg\varphi$ has a proof in T . By the definition of registry closure as theoremhood, neither $\Psi(\varphi)$ nor $\Psi(\neg\varphi)$ can be a closing state. Injectivity prevents identification with a different theorem-bearing sentence. $\square$

A new proof of incompleteness would require deriving an undecidable sentence from the registry axioms themselves; defining the interface to be the undecidable set only translates Godel's theorem.

49. Measurement closure and the Born-frequency bridge

Let the finite joint system-apparatus wake decompose into disjoint closure-compatible sectors Ω_i with finite positive Hausdorff measure $\mu_H(\Omega_i)$.

Theorem 49.1 (Frequency control with the full wake history). Let $(\mathcal{F}_j)$ be a filtration including the complete preparation and wake history through trial j . For sector i , put

$$Y_{j,i} = \mathbf{1}_{\{\text{outcome } j=i\}}, \quad p_{j,i} = \mathbb{E}(Y_{j,i} \mid \mathcal{F}_{j-1}), \quad f_N(i) = \frac{1}{N} \sum_{j=1}^N Y_{j,i}.$$

For every $N \geq 1$ and $\varepsilon > 0$,

$$\mathbb{P} \left(\left| f_N(i) - \frac{1}{N} \sum_{j=1}^N p_{j,i} \right| \geq \varepsilon \right) \leq 2e^{-2N\varepsilon^2}.$$

Almost surely, $f_N(i) - N^{-1} \sum_{j=1}^N p_{j,i} \rightarrow 0$. Successive preparations need not be independent.

Proof. Conditionally on $\mathcal{F}_{j-1}$ the indicator is Bernoulli with parameter $p = p_{j,i}$. Its centered log exponential moment is

$$\ell_p(t) = \log(1 - p + pe^t) - pt.$$

It has $\ell_p(0) = \ell'_p(0) = 0$ and $0 \leq \ell''_p(t) \leq 1/4$, so $\ell_p(t) \leq t^2/8$ for every real t . Iterated conditional expectation bounds the exponential moment of $\sum_{j \leq N} (Y_{j,i} - p_{j,i})$ by $e^{Nt^2/8}$. Markov's inequality with $t = 4\varepsilon$, and with its negative, gives the two tails. Their sum over N is finite for each positive rational ε . Borel–Cantelli gives almost-sure convergence, simultaneously for finitely many sectors. $\square$

Corollary 49.2 (Occupation-frequency identification). If the actual conditional occupations satisfy

$$\frac{1}{N} \sum_{j=1}^N p_{j,i} \longrightarrow \frac{\mu_H(\Omega_i)}{\sum_{\ell} \mu_H(\Omega_{\ell})} \quad \text{almost surely,}$$

then the observed frequencies have that limit. Independent preparations with the stated sector law are one sufficient case; wake-dependent preparations with the same limiting conditional occupations also qualify.

Proof. Add the vanishing difference of Theorem 49.1 to the displayed occupation limit. $\square$

Corollary 49.3 (Born identification criterion). For a complex coefficient readout (c_i) used to compare sector weights, if the finite realization derives

$$\mu_H(\Omega_i) = C|c_i|^2, \quad C > 0,$$

and the preceding occupation limit, then almost surely

$$f_N(i) \longrightarrow \frac{|c_i|^2}{\sum_j |c_j|^2}.$$

For normalized coefficients the limit is $|c_i|^2$.

Proof. Substitute into Corollary 49.2 and cancel C . $\square$

The fluctuation estimate retains the full wake filtration. The following finite construction supplies an occupation law directly from a retained deficit, and then computes its weights from the sector geometry.

Theorem 49.4 (Finite occupation law from a retained deficit). Fix $r \geq 2$ sectors. At event n , let $w_{n,i} \geq 0$ and $\sum_{i=1}^r w_{n,i} = 1$; these weights may depend on the complete earlier history. Start with $d_{0,i} = 0$. Choose an index i_n maximizing $d_{n-1,i} + w_{n,i}$, resolving ties by one fixed rule, and retain

$$d_{n,i} = d_{n-1,i} + w_{n,i} - \mathbf{1}_{\{i_n=i\}}. \quad (49.1)$$

For every finite event history,

$$-1 < d_{n,i} < r - 1, \quad \left| \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{\{i_n=i\}} - \frac{1}{N} \sum_{n=1}^N w_{n,i} \right| < \frac{r-1}{N}. \quad (49.2)$$

For constant weights w_i , the frequencies converge to w_i with the displayed deterministic error bound.

Proof. The deficits sum to zero at every step. Before selecting i_n , put $q_i = d_{n-1,i} + w_{n,i}$, so $\sum_i q_i = 1$. Inductively $d_{n-1,i} > -1$, and therefore every $q_i > -1$. The selected value satisfies $q_{i_n} \geq 1/r$, whence its successor $q_{i_n} - 1 \geq -(r-1)/r > -1$. Every unselected successor is $q_i > -1$. The zero sum then gives $d_{n,i} < r - 1$. Summing (49.1) gives $d_{N,i} = \sum_{n \leq N} w_{n,i} - \sum_{n \leq N} \mathbf{1}_{\{i_n=i\}}$, which proves the frequency bound. $\square$

Corollary 49.5 (Quadratic weights from two-dimensional sector geometry). Let Σ be a compact planar carrier with $0 < C = \mathcal{H}^2(\Sigma) < \infty$. Construct disjoint sector copies $\Omega_i = T_i(\Sigma)$, where

$$T_i(v) = a_i R_i v + b_i, \quad a_i \geq 0, \quad R_i^T R_i = I,$$

and not all a_i vanish. The orthogonal map R_i may preserve or reverse orientation. For the retained-deficit event (49.1) with the sector weights $w_i = \mathcal{H}^2(\Omega_i) / \sum_j \mathcal{H}^2(\Omega_j)$,

$$\left| f_N(i) - \frac{a_i^2}{\sum_j a_j^2} \right| < \frac{r-1}{N}.$$

In a complex amplitude coordinate c_i whose modulus is the geometric scale a_i , this is $f_N(i) \rightarrow |c_i|^2 / \sum_j |c_j|^2$.

Proof. Translations and orthogonal maps preserve Hausdorff measure. Scaling a covering by a_i multiplies every squared diameter by a_i^2 , and applying the inverse similarity gives the reverse inequality. Hence $\mathcal{H}^2(\Omega_i) = C a_i^2$ when $a_i > 0$; the zero-scale image has measure zero. Normalize these measures and apply Theorem 49.4. $\square$

This construction specifies both the sector geometry and the event allocation law. It establishes the occupation limit and the quadratic weight law for that realization, including orientation-reversing sector maps. An apparatus identification requires its occupied sector maps and successor rule to agree with these specified maps, or an independent proof that its frequency discrepancy obeys the same bound. The stochastic estimate of Theorem 49.1 applies separately to the actual conditional probabilities of a sampled realization.

Part VI

Finite-support incompressible transport

The finite-state and observation constructions now specialize to incompressible transport. The intrinsic carrier, channel handoff, and normalized transport bounds developed in Part I are retained, while the cochain realization specifies the momentum constraint and its exact readout. The shared vocabulary does not identify the fluid upper bound with the computational lower-bound questions of Part V.

Ordered states, admissible refinement, discrete second variation, persistent wake, and sign-separating carrier geometry remain primary in the fluid realization; continuum fields are observational readouts. Event endpoints are generated by a strictly increasing wake-realization map on the occupied finite history; laboratory time is defined only by synchronization with a second finite wake-realized reference history, and the symbol t is introduced afterward as an affine observational coordinate between completed event endpoints.

Section 57 constructs the finite-to-shadow bridge as an exact one-event cochain identity. Momentum is a closed face-flux cochain; resolved quadratic transport is defined by a finite

bilinear product map; retained wake stress is deposited from the incoming finite wake state before reconstruction, and the wake handoff then propagates the retained state into the successor event; pressure is the uniquely gauge-fixed multiplier enforcing the closed sector; forcing is a finite impulse rate; and viscosity is the discrete Hodge Laplacian with an explicit sign convention. Compatible Whitney reconstruction proves exact incompressibility and converts the event identity into an exact divergence-free continuum projected affine balance.

The general forced fluid law is identified as a wake-closed driven event system; a genuinely autonomous subclass is obtained only after the finite source/controller state generating impulse, event interval, wake handoff, and the entire occupied carrier/deposition geometry is included in the state. For the compatible periodic class, the finite event map and projected-affine evolution commute exactly, so $R_{M,k} = 0$; for noncompatible linear readouts the residual is bounded explicitly by operator commutators.

Section 58 gives a trace-compatible extension schema for orientable wall-bounded polyhedral domains and constructs its relative-cochain realization for impermeable walls. The periodic face-flux class is constructed in Section 57.8. Its momentum readout is distinguished from the complete inherited-wake accounting of Sections 52–56. A non-orientable Möbius wall is treated separately as an internal interface carrying twisted cochains; scalar orientation holonomy is distinguished from a two-sector carrier lift, and topology alone does not force nonzero circulation. Classical Navier–Stokes is studied as a smooth observational representation in Section 60, including nonzero retained stress whose divergence is absorbed into pressure.

The construction begins with intrinsic Hausdorff support and carrier realization, develops exact circulation readout, wake ancestry, and Wallis–Gregory bounds, and then treats nested three-body transport and the distinct D_H , p , and q scalings. Gaussian closure and continuum ontology are not taken as fundamental.

50. Category, admissible refinement, and the Hausdorff boundary

The finite-support setting uses the companion constructions [15–17]; the Hausdorff terminology follows [20].

Remark 50.1 (Finite-support convention). At each occupied event and fixed observational resolution M , the physical state is represented by finitely many cochain, wake, source, and geometric degrees of freedom. Finite support refers to finite occupation and finite encoding of admissible degrees of freedom; it does not require that the coordinate values themselves form a finite set.

The physical objects of the theory are finite ordered histories. Refinement is a relation among finite descriptions; it is not an assertion that a physical process executes infinitely

many infinitesimal steps. Let

$$\mathcal{R} = (S_n, C_n, R_n, T_n, h_n)_{n \geq 0} \quad (50.1)$$

be an admissible refinement system. Here S_n is a finite state set, C_n is a finite family of admissible three-point chains, $T_n : S_n \rightarrow \mathbb{R}$ is the ordering readout, and

$$R_n : S_n \longrightarrow 2^{S_{n+1}}, \quad R_n(p) \neq \emptyset \quad \text{for every } p \in S_n \quad (50.2)$$

is the set-valued descendant map. The number $h_n > 0$ is the observational resolution of level n , with $h_n \downarrow 0$ in the comparison family.

Definition 50.2 (Coherent refinement branch). A coherent branch descending from $p_0 \in S_0$ is a sequence

$$p = (p_0, p_1, p_2, \dots), \quad p_{n+1} \in R_n(p_n). \quad (50.3)$$

The set of all coherent infinite branches is denoted $\partial\mathcal{R}$. The boundary is a mathematical completion of the finite refinement family; every occupied physical history used below remains finite.

Definition 50.3 (Primary second variation). For $c = (p_{i-1}, p_i, p_{i+1}) \in C_n$, define

$$\Delta_{T_n}^2(c) := T_n(p_{i+1}) - 2T_n(p_i) + T_n(p_{i-1}). \quad (50.4)$$

This finite second variation is the primary sign invariant.

Definition 50.4 (Persistent sign branch). For $\sigma \in \{-1, +1\}$ let $B_\sigma \subset \partial\mathcal{R}$ consist of coherent branches p for which there are $N_\sigma < \infty$ and chains $c_n \in C_n$ containing p_n such that

$$\text{sgn } \Delta_{T_n}^2(c_n) = \sigma \quad \text{for every } n \geq N_\sigma. \quad (50.5)$$

Thus persistence is tested along one coherent descendant branch rather than by selecting unrelated descendants at successive levels.

Fix $0 < \rho < 1$. For distinct branches $p, q \in \partial\mathcal{R}$, let

$$N(p, q) := \min\{n \geq 0 : p_n \neq q_n\}, \quad d_{\text{adm}}(p, q) := \rho^{N(p, q)}, \quad (50.6)$$

and put $d_{\text{adm}}(p, p) = 0$.

Proposition 50.5 (Refinement ultrametric). The function d_{adm} is an ultrametric:

$$d_{\text{adm}}(p, r) \leq \max\{d_{\text{adm}}(p, q), d_{\text{adm}}(q, r)\}. \quad (50.7)$$

If the refinement system is finitely branching and has no terminal descendant, $\partial\mathcal{R}$ is compact.

Proof. If p and q agree through level $m - 1$, and q and r agree through level $m - 1$, then p and r also agree through level $m - 1$. Hence $N(p, r) \geq \min\{N(p, q), N(q, r)\}$, which is equivalent to (50.7). Finite branching makes the space totally bounded: for each n there are only finitely many length- n prefixes, whose cylinders are balls of radius ρ^n . A Cauchy sequence stabilizes at every finite prefix and therefore determines a coherent branch, so the space is complete. Complete plus totally bounded implies compactness.

Definition 50.6 (Refinement-stable sign carrier). The limiting sign-separating carrier is

$$L_\infty := \overline{B_+}^{d_{\text{adm}}} \cap \overline{B_-}^{d_{\text{adm}}}. \quad (50.8)$$

Thus every admissible neighborhood of a branch in L_∞ contains sufficiently refined histories from both persistent sign classes. This is the intrinsic finite-refinement separatrix.

Definition 50.7 (Intrinsic Hausdorff support). The Hausdorff dimension of the carrier is

$$D_H := \dim_H^{d_{\text{adm}}}(L_\infty), \quad \chi_H := D_H - 1. \quad (50.9)$$

No Euclidean embedding is required for this definition.

Proposition 50.8 (Regular refinement calibration). If every carrier cylinder has exactly $b \geq 2$ carrier descendants and every refinement contracts admissible diameter by the same factor ρ , then

$$D_H = \frac{\log b}{-\log \rho}. \quad (50.10)$$

Proof. At level n the carrier is covered by b^n cylinders of diameter ρ^n . The critical exponent s for which $b^n(\rho^n)^s = (b\rho^s)^n$ changes from divergence to decay is determined by $b\rho^s = 1$, giving (50.10). Assigning mass b^{-n} to every level- n cylinder gives the matching lower bound.

The notation χ_H is deliberately geometric. It is not identified with a Kolmogorov exponent, a transport exponent, or a nonextensive statistical index.

51. Carrier realization and canonical anticonformal form

The sign data alone do not manufacture a Euclidean plane. The finite theory therefore includes an explicit carrier realization map rather than hiding this step in coordinates.

Definition 51.1 (Transverse carrier realization). Let $\mathcal{C}_{\text{trans}}$ denote tuples

$$(E, g, J, L^+, L^-) \quad (51.1)$$

where (E, g) is an oriented two-dimensional real inner-product space, $J : E \rightarrow E$ satisfies $J^2 = -I$, and $L^+, L^- \subset E$ are transverse one-dimensional subspaces. A carrier realization is a map

$$\Phi_{\text{car}} : L_{\infty} \longrightarrow \mathcal{C}_{\text{trans}} \quad (51.2)$$

that assigns the two persistent sign sectors to L^+ and L^- , respectively.

Definition 51.2 (Sign involution). For $\Phi_{\text{car}}(p) = (E_p, g_p, J_p, L_p^+, L_p^-)$, define $C_p : E_p \rightarrow E_p$ by

$$C_p|_{L_p^+} = +I, \quad C_p|_{L_p^-} = -I. \quad (51.3)$$

The realization is balanced when $L_p^+ \perp L_p^-$, $J_p L_p^+ = L_p^-$, and J_p is a g_p -isometry, with the complex orientation of J_p . The metric g_J of Proposition 6.3 constructs these properties from arbitrary positive initial metric data.

Theorem 51.3 (Canonical carrier normal form). For a balanced carrier,

$$C_p^2 = I, \quad C_p^* = C_p, \quad C_p J_p = -J_p C_p. \quad (51.4)$$

In an orthonormal sign basis,

$$C_p \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (51.5)$$

After the orthogonal 45° basis change

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (51.6)$$

one has

$$U^T C_p U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: C. \quad (51.7)$$

Hence a retained aperture factor $0 < \kappa_y \leq 1$ gives

$$H_y = \kappa_y C, \quad H_y^T H_y = \kappa_y^2 I. \quad (51.8)$$

Proof. The first two identities follow directly from the ± 1 action on the orthogonal sign lines. Because J interchanges the two oriented sign lines, applying C before or after J reverses the sign, giving anticommutation. Equations (51.5)–(51.7) are direct matrix multiplication, and the final norm identity follows from $C^T C = I$.

The algebraic realization has a concrete choice: assign the same $E = \mathbb{R}^2$, coordinate sign lines, standard quarter-turn J , and Euclidean metric at every carrier point. More generally Proposition 6.3 constructs a compatible metric for any specified algebraic sign exchange. The physical identification of a chosen carrier remains separate from this proved mathematical existence.

52. Exact wake handoff and channel recombination

Let a_j be the complete retained wake arriving at event j . For fixed source and recorded history, let B_j denote the event response on the wake leg, let $P_{L,j+1}$ be the longitudinal output projection, and let $K_{j+1,j}$ be the ordered transfer map into the next event. Define

$$M_j := K_{j+1,j} P_{L,j+1} B_j. \quad (52.1)$$

Proposition 52.1 (Source-to-successor handoff). The incoming wake satisfies

$$a_{j+1} = M_j a_j, \quad a_N = M_{N-1} \cdots M_0 a_0. \quad (52.2)$$

The map M_j may depend nonlinearly on the source and recorded history even when its action on the incoming wake leg is linear.

Proof. The first identity is the definition of the wired event. Repeated finite substitution gives the second. No limiting operation is used.

52.1. Anticonformal source law and retained conditioning

The response and successor maps in (52.1) are supplied by the anticonformal construction [18], equations (86), (94)–(95), and (144). For a completed event word $k_0 < \cdots < k_{r-1}$, source pairs $(\bar{u}_j, u_j)$, and nonlinear degrees m_j , their identification is

$$B_j a = \mathcal{E}_{k_j}^{[m_j]}[\bar{u}_j, u_j; a \mid Z_{k_j}^{\text{rec}}], \quad (52.3)$$

$$K_{j+1,j} = K_{k_{j+1}k_j}[Z_{k_{j+1}}^{\text{rec}}, Z_{k_j}^{\text{rec}}], \quad (52.4)$$

$$a_{j+1} = K_{k_{j+1}k_j} \mathcal{E}_{L,k_j}^{[m_j]}[\bar{u}_j, u_j; a_j \mid Z_{k_j}^{\text{rec}}]. \quad (52.5)$$

Here $\mathcal{E}_L = P_L \mathcal{E}$, and the kernel is generated by the ordered finite composition in equation (27) of that paper. Its source slots remain distinct from the incoming wake slot. The source response coefficients are generated by its equations (88)–(89):

$$\mathcal{A}_n^{[0]} = G_{A,n} E_{AI,n} u_n, \quad \mathcal{A}_n^{[m+1]} = g_n G_{A,n} \sum_{a+b+c=m} (\mathcal{A}_n^{[a]})^* \mathcal{A}_n^{[b]} \mathcal{A}_n^{[c]}. \quad (52.6)$$

The event data $(g_n, G_{A,n}, E_{AI,n})$ are those declared by the source ledger. In particular, the successor law is the source's open vertex and ordered kernel evaluated on the occupied history.

The retained reciprocation data are explicitly

$$Z_n^{\text{rec}} = (Z_n, D_n^{\text{GW}}), \quad D_n^{\text{GW}} = W_{2c_n,n} - 2G_{c_n,n}, \quad (52.7)$$

where c_n is the completed-flip counter. Both the vertex and the kernel are conditioned on this state. Thus the Gregory–Wallis remnant is already present in the next event’s data; it is not inserted again as an independent source. A released channel that later returns is retained in the live state as specified in Section 53.1. The longitudinal word (52.5) alone does not contain such an extra return edge.

For a positive source reference flux, source normalization gives

$$\hat{B}_j = \frac{B_j}{\Phi_{k_j}^{\text{ref}}(\bar{u}_j, u_j)^{m_j+1}}.$$

The numerator and denominator acquire the same factor $|c|^{2(m_j+1)}$ under nonzero source scaling, so $\hat{B}_j$ is unchanged. This proves normalization invariance. A positive-flux estimate for the occupied response is instead obtained from its channel budget or the actual-work identity below. The source’s separate aperture budget is treated in Theorem 54.1.

The real cochain formulas use the underlying real spaces of any complex flux spaces. An anti-linear map is real-linear there, so the real metric transpose formulas below also apply to an anticonformal event.

Write the complete wake as

$$a_j = c_j + r_j, \quad (52.8)$$

where $c_j = P_j a_j$ lies in a chosen canonical subspace and $r_j = Q_j a_j$, $Q_j = I - P_j$, is the retained residual.

Theorem 52.2 (Residual re-entry). The exact block recurrence is

$$c_{j+1} = P_{j+1} M_j c_j + P_{j+1} M_j r_j, \quad (52.9)$$

$$r_{j+1} = Q_{j+1} M_j c_j + Q_{j+1} M_j r_j. \quad (52.10)$$

In particular, $P_{j+1} M_j r_j$ is inherited residual content that re-enters the next canonical component.

Proof. Apply P_{j+1} and Q_{j+1} to $a_{j+1} = M_j(c_j + r_j)$.

Proposition 52.3 (Complete channel identity). For an orthogonal output decomposition $I = P + Q$ and arbitrary input components x, y ,

$$\|M(x + y)\|^2 = \|PM(x + y)\|^2 + \|QM(x + y)\|^2, \quad (52.11)$$

$$\langle Mx, My \rangle = \langle PMx, PMy \rangle + \langle QMx, QMy \rangle. \quad (52.12)$$

Thus a canonical channel may increase by receiving residual content while the complete norm remains controlled.

52.2. Successive wakes encapsulate the earlier remnants

For the occupied sequence define the four restricted maps

$$A_j = P_{j+1}M_j|_{\text{Ran } P_j}, \quad B_j = P_{j+1}M_j|_{\text{Ran } Q_j}, \quad C_j = Q_{j+1}M_j|_{\text{Ran } P_j}, \quad D_j = Q_{j+1}M_j|_{\text{Ran } Q_j}.$$

These are blocks of the given handoff, not additional dynamics. Write $D_{n:j} = D_{n-1} \cdots D_j$ for $n > j$ and $D_{j:j} = I$.

Theorem 52.4 (Exact inherited-remnant memory). At every finite event,

$$r_n = D_{n:0}r_0 + \sum_{j=0}^{n-1} D_{n:j+1}C_jc_j, \quad (52.13)$$

$$c_{n+1} = A_nc_n + B_nD_{n:0}r_0 + \sum_{j=0}^{n-1} B_nD_{n:j+1}C_jc_j. \quad (52.14)$$

Thus the next canonical component contains the transported initial remnant and the remnants generated at every earlier event. They are already encapsulated in r_n when the next handoff is performed.

Proof. The residual block equation is $r_{j+1} = C_jc_j + D_jr_j$. Substitution proves (52.13) by induction, with the empty sum at $n = 0$. Insert that identity into $c_{n+1} = A_nc_n + B_nr_n$ to obtain (52.14). The coefficients may depend on the recorded history: each is evaluated on the same occupied history throughout this finite substitution. No independence assumption is made about the summands. $\square$

The term B_nr_n is an internal transfer of inherited content. Treating it as a new, independently supplied load at every return counts the same ancestry again. Eliminating r_n produces a law with memory for c_n ; it does not justify an autonomous update depending only on the present c_n . For fixed controller data, such a one-step canonical update exists on the full direct-sum state space exactly when $B_n = 0$: two states with the same c_n and residual difference v have successor difference B_nv . On a restricted occupied state set the exact criterion is constancy on its observed fibers, as in Theorem 38.1. The retained state must also include any controller or history data on which the next maps depend. Along a specified history the displayed block identities remain exact whether or not the canonical observation closes.

53. Finite flux balance

At event j set

$$b_j = B_ja_j, \quad \ell_j = P_Lb_j, \quad t_j = P_\perp b_j, \quad x_j = P_Xb_j, \quad a_{j+1} = K_j\ell_j.$$

Define the event and transport changes

$$e_j^{\text{event}} := \|B_j a_j\|_{\text{out}}^2 - \|a_j\|^2, \quad (53.1)$$

$$e_j^{\text{tr}} := \|K_j \ell_j\|^2 - \|\ell_j\|^2. \quad (53.2)$$

These are changes evaluated on the propagated state, not independent random forcing variables.

Theorem 53.1 (Occupied-history flux identity). For every finite wired sequence,

$$\|a_{j+1}\|^2 + \|t_j\|^2 + \|x_j\|^2 = \|a_j\|^2 + e_j^{\text{event}} + e_j^{\text{tr}}, \quad (53.3)$$

and consequently

$$\|a_N\|^2 + \sum_{j=0}^{N-1} (\|t_j\|^2 + \|x_j\|^2) = \|a_0\|^2 + \sum_{j=0}^{N-1} (e_j^{\text{event}} + e_j^{\text{tr}}). \quad (53.4)$$

Proof. Orthogonality gives $\|B_j a_j\|_{\text{out}}^2 = \|\ell_j\|^2 + \|t_j\|^2 + \|x_j\|^2$. Insert (53.1) and (53.2). Summing over j telescopes the internal incoming/outgoing wake norms.

53.1. Accounting across successive returns

Equation (53.4) counts t_j and x_j as outputs leaving the selected longitudinal chain. If one of those outputs is wired into a later event, it must instead remain in the live retained state until that event consumes it. Its earlier value is not simultaneously an exported loss and a newly supplied input. A stored record of a channel and the channel itself are likewise not two physical loads.

Theorem 53.2 (Finite live-channel balance). Consider a finite ordered event history. Give each carried channel its squared norm q_e , expressed in its transported metric. Each internal channel is an output of one event and an input of exactly one later event; a physical split is represented by its complete output channels. Suppose the local ledger is

$$\sum_{e \in \text{out}(v)} q_e - \sum_{e \in \text{in}(v)} q_e = w_v.$$

Then, at every cut after N events,

$$\mathcal{E}_N + \mathcal{E}_{\text{export},N} = \mathcal{E}_0 + \sum_{v < N} w_v, \quad (53.5)$$

where $\mathcal{E}_N$ sums all channels still retained, including delayed remnants, and the export term contains only channels that leave this history without later re-entry. In particular, lossless completed events and isometric wiring preserve the complete live energy when there are no exports.

Proof. Sum the local identities. Every channel consumed after being produced appears once with each sign and cancels. The unmatched inputs are the initial channels; the unmatched outputs are precisely the live channels and genuine exports. This argument includes a remnant that waits through several events before it returns. For a multi-input event its input norm is the direct-sum norm; its local identity must include all output channels and their recombination terms. $\square$

The quantity w_v records actual event work or dissipation, not residual re-entry. The next section proves zero local work for the completed aperture. An arbitrary event response B_j can still have nonzero work; retaining its history does not by itself set that work to zero.

Proposition 53.3 (Exact work of the actual occupied handoff). Let H_j be the positive metric on the complete incoming wake at event j , including the geometric area factor when that factor is used. For the actual map $M_j = K_{j+1,j}P_{L,j+1}B_j$ define

$$W_j = M_j^T H_{j+1} M_j - H_j, \quad (53.6)$$

$$\Phi_0 = I, \quad \Phi_j = M_{j-1} \cdots M_0 \quad (j \geq 1). \quad (53.7)$$

Then, along every finite occupied history,

$$\Phi_N^T H_N \Phi_N = H_0 + \sum_{j=0}^{N-1} \Phi_j^T W_j \Phi_j, \quad (53.8)$$

and consequently

$$\|a_N\|_N^2 = \|a_0\|_0^2 + \sum_{j=0}^{N-1} a_j^T W_j a_j. \quad (53.9)$$

No per-event contraction is required for these identities. In particular, an upper bound on the cumulative signed work bounds the complete wake even when individual events increase its norm.

Proof. Since $\Phi_{j+1} = M_j \Phi_j$,

$$\Phi_{j+1}^T H_{j+1} \Phi_{j+1} - \Phi_j^T H_j \Phi_j = \Phi_j^T W_j \Phi_j.$$

Sum and evaluate the resulting quadratic form at a_0 . The exact recurrence gives $a_j = \Phi_j a_0$. History-dependent coefficients are evaluated on that same history; the identity is not an assertion about a different trajectory. $\square$

Writing $a_j = c_j + r_j$ in (53.9) retains the mixed term $2c_j^T W_j r_j$ as well as both diagonal terms. Thus this test includes remnant return and its signed interaction with the canonical channel. For the actual factorization its work matrix is explicitly

$$W_j = B_j^T P_{L,j+1}^T K_{j+1,j}^T H_{j+1} K_{j+1,j} P_{L,j+1} B_j - H_j.$$

If some channels leave the selected map and later return, apply the same identity to the enlarged live-channel map of Theorem 53.2. This is the matrix that must be evaluated to establish a bound for the actual wiring. The completed-aperture construction below supplies a local zero-work identity wherever those completed maps are the specified event factors.

54. Completed aperture and the 1:3 transverse frame

The carrier operator H_y of (51.8) is nonexpansive:

$$\|H_y \xi\| = \kappa_y \|\xi\| \leq \|\xi\|. \quad (54.1)$$

If an aperture map A satisfies $A^\dagger A \leq I$, define

$$D = (I - A^\dagger A)^{1/2}. \quad (54.2)$$

For the complete output $Au \oplus Du$,

$$\|u\|^2 = \|P_L Au\|^2 + \|P_\perp Au\|^2 + \|Du\|^2. \quad (54.3)$$

Nothing is discarded by the aperture split.

Theorem 54.1 (Complete retained flux for the source's passive apertures). For the admissible compression in equations (104)–(108) of [18], write

$$A_n = P_{n+1}^A F_n P_n^A, \quad D_n = (I - A_n^{\dagger\Phi} A_n)^{1/2}, \quad \mathcal{J}_n v = (P_L A_n v, P_\perp A_n v, D_n v).$$

In the source's passive-aperture class $A_n^{\dagger\Phi} A_n \leq I$, every occupied incoming state satisfies

$$\|\mathcal{J}_n v\|^2 = \|v\|_\Phi^2. \quad (54.4)$$

For every finite history of these completed events, with isometric channel wiring and every delayed or returning output retained exactly once, the complete live-channel energy obeys

$$\mathcal{E}_N + \mathcal{E}_{\text{export},N} = \mathcal{E}_0. \quad (54.5)$$

This statement includes arbitrary history dependence of the actual compressions. It does not require A_n to be a scalar multiple of a self-adjoint involution.

Proof. Orthogonality of the admissible outputs and the definition of D_n give

$$\mathcal{J}_n^\dagger \mathcal{J}_n = A_n^{\dagger\Phi} (P_L + P_\perp) A_n + D_n^\dagger D_n = I.$$

This identity is applied to the entire occupied input v , including all remnants already arriving there. Its quadratic form therefore retains their mixed terms. Waiting channels pass unchanged through a cut, and an isometric transfer preserves the norm assigned to its carried channel. Theorem 53.2 now telescopes the local zero-work identities. An exported channel is counted in the export term only if it does not later return. Any multi-input recombination is part of the completed local map whose input and output norms enter that identity. $\square$

The source's Assumption 7.1 specifies the passive-aperture class. Theorem 54.3 below supplies its explicit multi-input completion, and Corollary 54.4 derives the contraction inequality for orthogonal compressions of transported isometries. Additional physical incoming channels contribute their flux to the input side of (54.5); a stored record or a returning remnant is not such a new input. Transport satisfying Proposition 56.1 supplies the required isometry in its geometric norm. For the open homogeneous response (52.3), the actual response and transport work remain those in Proposition 53.3; no equality between each homogeneous coefficient and the entire completed aperture is assumed.

For three unit directions e_1, e_2, e_3 separated by 120° in the transverse carrier,

$$\sum_{a=1}^3 e_a e_a^T = \frac{3}{2} I, \quad (54.6)$$

so

$$\|v\|^2 = \frac{2}{3} \sum_{a=1}^3 |\langle v, e_a \rangle|^2. \quad (54.7)$$

This is a tight-frame identity and therefore controls arbitrary Ansio/Iso amplitude distributions; it does not impose equal amplitudes on the three channels.

54.1. A completed aperture with repeated remnant return

The identities above apply to the complete output, not merely its present longitudinal observation. For the stated carrier aperture $H_n = \kappa_n C_n$, where $C_n^* = C_n$ and $C_n^2 = I$, put $d_n = \sqrt{1 - \kappa_n^2}$. Then

$$\mathcal{J}_n u = (P_L H_n u, P_\perp H_n u, d_n u), \quad \|\mathcal{J}_n u\|^2 = \|u\|^2.$$

This proves the complete local identity directly from the given aperture. All future-relevant outputs are carried forward in the live state of Theorem 53.2. The frame coefficients in (54.7) use their factor $2/3$; counting them as three unweighted copies would change the represented norm.

For a finite state that allows an existing remnant to return at every event, the same aperture has the following explicit completion on two copies of its carrier:

$$\mathcal{S}_n = \begin{pmatrix} H_n & d_n I \\ d_n I & -H_n \end{pmatrix}, \quad \begin{pmatrix} \hat{c}_{n+1} \\ \hat{r}_{n+1} \end{pmatrix} = \mathcal{S}_n \begin{pmatrix} c_n \\ r_n \end{pmatrix}. \quad (54.8)$$

Theorem 54.2 (Completed-aperture norm and residual return). For every incoming pair, including nonzero inherited r_n ,

$$\|H_n c_n + d_n r_n\|^2 + \|d_n c_n - H_n r_n\|^2 = \|c_n\|^2 + \|r_n\|^2. \quad (54.9)$$

Consequently $\mathcal{S}_n^* \mathcal{S}_n = I$. A finite composition of these completed apertures, orthogonal channel splits, and isometric channel wiring preserves the complete norm. Retaining only a subset of the final outputs gives the corresponding nonexpansive readout.

Proof. The two cross terms are $2d_n \langle H_n c_n, r_n \rangle$ and $-2d_n \langle c_n, H_n r_n \rangle$ and cancel because $H_n^* = H_n$. Also $H_n^* H_n = \kappa_n^2 I$ and $d_n^2 + \kappa_n^2 = 1$. This proves (54.9). Composition preserves the identity; an orthogonal final projection can only decrease the norm. The identity is pointwise, so κ_n may be selected by the occupied history without changing this norm calculation. $\square$

For example, choose $\kappa = 3/5$, $d = 4/5$, and the exchange matrix C from (51.7). Starting with $(c_0, r_0) = (0, r_0)$ gives $(c_1, r_1) = (4r_0/5, -3Cr_0/5)$. The canonical norm increases from zero, but the complete norm is unchanged. A second identical handoff returns the full pair to $(0, r_0)$ because $\mathcal{S}^2 = I$. The visible increase is supplied by the inherited residual and is not an independent energy increment.

Equation (54.8) gives the balanced involutive aperture. The following completion treats a general passive compression with an already occupied return channel. For a specified source wiring, its factorization into these complete maps determines the norm law; the actual-work identity (53.5) accounts for any additional source work.

Theorem 54.3 (General passive aperture with a returning remnant). Let $A : H \rightarrow K$ be a contraction between finite-dimensional positive-flux spaces. All adjoints and norms refer to those metrics. Put

$$D_H = (I_H - A^* A)^{1/2}, \quad D_K = (I_K - A A^*)^{1/2}.$$

The complete event on an incoming canonical state $c \in H$ and an already retained return $r \in K$ is

$$\mathcal{S}_A(c, r) = (Ac + D_K r, D_H c - A^* r). \quad (54.10)$$

For every incoming pair,

$$\|Ac + D_K r\|_K^2 + \|D_H c - A^* r\|_H^2 = \|c\|_H^2 + \|r\|_K^2. \quad (54.11)$$

Hence arbitrary history-dependent choices of A , composed with isometric channel wiring, preserve the complete retained norm. Every canonical/residual mixed term is included.

Proof. The identity $(A A^*) A = A (A^* A)$ implies $p(A A^*) A = A p(A^* A)$ for every polynomial p . Polynomial approximation to $\sqrt{1-t}$ on $[0, 1]$ gives $D_K A = A D_H$, and taking adjoints gives $A^* D_K = D_H A^*$. Expanding the left side of (54.11), the c terms sum to $\langle c, (A^* A + D_H^2) c \rangle = \|c\|^2$, and the r terms sum to $\langle r, (D_K^2 + A A^*) r \rangle = \|r\|^2$. The cross terms cancel by the intertwining identity. Composition proves the history statement. $\square$

This is the classical contraction completion, also called a Julia or Halmos dilation [19], applied here to the occupied return channel. It supplies an explicit multi-input event for the general passive aperture, including apertures that are not scalar multiples of a self-adjoint involution. For $A = H_n = \kappa_n C_n$ with $C_n^* = C_n$ and $C_n^2 = I$, both defects equal $\sqrt{1 - \kappa_n^2} I$ and (54.10) becomes (54.8). Orthogonal splitting of its first output into longitudinal and transverse channels gives the three-channel ledger.

Corollary 54.4 (Passivity derived from geometric compression). Let $T : H \rightarrow K$ preserve the transported complete-state norm, and let P_H, P_K be orthogonal admissibility projections in the corresponding metrics. Then the actual compression $A = P_K T P_H$ satisfies $A^* A \leq I$, and Theorem 54.3 applies.

Proof. For every v , $\|P_K T P_H v\| \leq \|T P_H v\| = \|P_H v\| \leq \|v\|$. This is the quadratic-form inequality $A^* A \leq I$. Proposition 56.1 supplies the required isometry when the norm is transported by the stated carrier-area law. $\square$

55. Exact Wallis–Gregory remainder and a uniform retained-state bound

The finite products and sums below use the Wallis and Gregory constructions [21,22].

The finite arithmetic used to control the retained scalar remainder is stated independently of the fluid interpretation. Define

$$w_j = \frac{4(j+1)^2}{(2j+1)(2j+3)}, \quad W_j = \prod_{h=0}^j w_h, \quad (55.1)$$

$$g_m = \frac{1}{4m+1} - \frac{1}{4m+3}, \quad G_m = \sum_{r=0}^m g_r, \quad (55.2)$$

and

$$\rho_m = W_{2m} - 2G_m, \quad \alpha_m = w_{2m-1} w_{2m}. \quad (55.3)$$

The symbol α_m is used here so that the statistical index q below remains unambiguous. Put

$$f_m = 2G_{m-1}(\alpha_m - 1) - 2g_m, \quad P_m = \prod_{h=1}^m \alpha_h, \quad P_0 = 1. \quad (55.4)$$

Since $P_m = W_{2m}/W_0$ and $W_0 = 4/3$,

$$P_m \longrightarrow \frac{3\pi}{8}. \quad (55.5)$$

Lemma 55.1 (Exact scalar recurrence). For $m \geq 1$,

$$\rho_m = \alpha_m \rho_{m-1} + f_m, \quad \rho_0 = 0. \quad (55.6)$$

Moreover

$$f_1 = \frac{8}{525}, \quad f_m < 0 \ (m \geq 2), \quad 0 < \rho_m \leq \frac{8}{525}. \quad (55.7)$$

Proof. The recurrence follows by inserting $W_{2m} = \alpha_m W_{2m-2}$ and $G_m = G_{m-1} + g_m$ into $\rho_m = W_{2m} - 2G_m$. For the sign, set $x_m = 4m + 1$. Direct algebra gives

$$\alpha_m - 1 = \frac{2x_m^2 + 1}{x_m^2(x_m^2 - 4)}, \quad b_m := \frac{g_m}{\alpha_m - 1} = \frac{2x_m(x_m - 2)}{2x_m^2 + 1}. \quad (55.8)$$

The function $b(x) = 2x(x - 2)/(2x^2 + 1)$ is increasing for $x \geq 5$ because

$$b'(x) = \frac{4(2x - 1)(x + 1)}{(2x^2 + 1)^2} > 0.$$

At $m = 1$ direct evaluation gives $f_1 = 8/525$. At $m = 2$, $G_1 = 76/105 < 126/163 = b_2$. For $m \geq 3$, the finite alternating upper bound gives $G_{m-1} < \pi/4 < 263/315 < 286/339 = b_3 \leq b_m$. Since

$$f_m = 2(\alpha_m - 1)(G_{m-1} - b_m),$$

we obtain $f_m < 0$ for $m \geq 2$. Also

$$\rho_m - \rho_{m-1} = (\alpha_m - 1)(W_{2m-2} - 2b_m). \quad (55.9)$$

At $m = 2$, $W_2 = 256/175 < 252/163 = 2b_2$; for $m \geq 3$, $W_{2m-2} < \pi/2 < 526/315 < 572/339 = 2b_3 \leq 2b_m$. Thus ρ_m decreases after $m = 1$. Finally $W_{2m} \rightarrow \pi/2$ and $2G_m \rightarrow \pi/2$, hence $\rho_m \rightarrow 0$. Positivity and (55.7) follow.

Theorem 55.2 (Exact transported variation). For every $N \geq 1$,

$$\sum_{m=1}^N \frac{f_m}{P_m} = \frac{\rho_N}{P_N}, \quad (55.10)$$

$$\sum_{m=1}^N \frac{|f_m|}{P_m} = \frac{1}{36} - \frac{\rho_N}{P_N}, \quad (55.11)$$

$$\sum_{m=1}^N \frac{P_N}{P_m} |f_m| = \frac{P_N}{36} - \rho_N < \frac{\pi}{96}. \quad (55.12)$$

The supremum of the last quantity is $\pi/96$.

Proof. Divide (55.6) by $P_m = \alpha_m P_{m-1}$ and telescope. The first normalized increment is

$$\frac{8/525}{192/175} = \frac{1}{72},$$

while all later increments are negative. Hence the absolute sum equals twice the first increment minus the signed sum, giving (55.11). Multiply by P_N and use (55.5) and $\rho_N \rightarrow 0$.

Theorem 55.3 (Uniform complete-state bound). Use the full retained-state norm at each event, denoted $\|\cdot\|_m$; this includes the case of one fixed norm. Suppose the retained state obeys

$$x_m = \alpha_m U_m(x_{m-1}) + f_m e_m, \quad \|U_m(x_{m-1})\|_m \leq \|x_{m-1}\|_{m-1}, \quad \|e_m\|_m \leq E_w. \quad (55.13)$$

Then

$$\|x_N\|_N \leq P_N \|x_0\|_0 + E_w \left(\frac{P_N}{36} - \rho_N \right) \leq \frac{3\pi}{8} \|x_0\|_0 + \frac{E_w \pi}{96}. \quad (55.14)$$

Proof. From (55.13), $\|x_m\|_m \leq \alpha_m \|x_{m-1}\|_{m-1} + E_w |f_m|$. Divide by P_m , sum, and use (55.12).

Corollary 55.4 (Wallis–Gregory control of the inherited wake). For the recurrence (55.13), let U_m be the occupied handoff built from the completed apertures and isometric wiring of Theorem 54.2, with every returning residual retained. Use the transported norm proved in Proposition 56.1 when the carrier geometry changes. Then the nonexpansive inequality in (55.13) follows from those local identities, and (55.14) holds. No separate bound on each residual return is needed.

Proof. The complete local norm identities telescope by Theorem 53.2; the geometric transport preserves the normalized norm by Proposition 56.1. Hence $\|U_m(x)\|_m \leq \|x\|_{m-1}$. Apply Theorem 55.3. $\square$

Here x_m contains the inherited canonical and residual channels. The term $f_m e_m$ is the explicitly specified Wallis–Gregory remainder contribution; the terms $B_n r_n$ in (52.14) belong to U_m . The norm inequality allows the displayed remainder direction to depend on the history; it does not assume statistically independent forcing. What is essential is that the same remainder is not entered twice in the ledger. The scalar identities determine the coefficients and their variation. The following two constructions specify how the remainder is generated from the retained state: co-transport of its initial packet and readout of the complete incoming state.

Theorem 55.5 (Co-transported remainder packets). There is a sharper identity when the remainder direction is itself inherited through the same linear wake leg. Let $\mathcal{U}_m$ denote the completed handoff evaluated on the occupied history, and set

$$e_m = \mathcal{U}_m e_{m-1}, \quad x_m = \alpha_m \mathcal{U}_m x_{m-1} + f_m e_m.$$

Then, with $\mathcal{U}_{N:0} = \mathcal{U}_N \cdots \mathcal{U}_1$,

$$x_N = \mathcal{U}_{N:0} (P_N x_0 + \rho_N e_0), \quad e_N = \mathcal{U}_{N:0} e_0. \quad (55.15)$$

In particular, the completed norm identity proves $\|e_N\|_N = \|e_0\|_0$ and

$$\|x_N\|_N \leq P_N \|x_0\|_0 + \rho_N \|e_0\|_0 \leq \frac{3\pi}{8} \|x_0\|_0 + \frac{8}{525} \|e_0\|_0. \quad (55.16)$$

Thus both the handoff norm and the packet bound are proved for this co-transported subclass from the completed maps and the initial packet.

Proof. The formula holds at $N = 0$ since $P_0 = 1$ and $\rho_0 = 0$. If it holds at $N - 1$, linearity on the occupied wake leg gives

$$x_N = \mathcal{U}_{N:0}(\alpha_N P_{N-1} x_0 + (\alpha_N \rho_{N-1} + f_N) e_0).$$

Use $P_N = \alpha_N P_{N-1}$ and the exact scalar remainder recurrence (55.6). Norm preservation proves the packet equality and the first inequality; $0 \leq \rho_N \leq 8/525$ gives the second. History dependence of the maps does not change the proof: the same evaluated map transports both terms at each event. $\square$

This identity retains the signed cancellations in the successive remainder ledger. No new packet direction is supplied at later events. If an occupied handoff rotates different remainder contributions differently, the exact co-transport identity need not apply, while Corollary 55.4 still gives its absolute-variation bound. The next theorem treats changing residual directions by reading the remainder from the complete inherited state at each event.

Theorem 55.6 (Inherited remainder control from the initial state). At event m , let $\mathcal{U}_m$ carry the complete incoming state into the next positive-flux space with $\|\mathcal{U}_m x\|_m \leq \|x\|_{m-1}$. Its choice may depend on the occupied history. Define

$$y_m = \mathcal{U}_m x_{m-1}, \quad e_m = \mathcal{R}_m y_m, \quad x_m = \alpha_m y_m + f_m e_m,$$

where $\mathcal{R}_m$ is an inherited-state readout satisfying $\|\mathcal{R}_m y\|_m \leq \|y\|_m$. Then no independent packet bound is required:

$$\|x_N\|_N \leq P_N \prod_{m=1}^N \left(1 + \frac{|f_m|}{\alpha_m}\right) \|x_0\|_0 \leq \frac{3\pi}{8} e^{\pi/96} \|x_0\|_0. \quad (55.17)$$

If $\mathcal{R}_m = Q_m$ is an orthogonal residual projection, the sharper bound is

$$\|x_N\|_N \leq \frac{73}{72} P_N \|x_0\|_0 \leq \frac{73\pi}{192} \|x_0\|_0, \quad N \geq 1. \quad (55.18)$$

The projections may rotate or change rank at every event and need not commute with any handoff. The last constant is sharp for this class.

Proof. The contraction inequalities give $\|x_m\|_m \leq (\alpha_m + |f_m|)\|x_{m-1}\|_{m-1}$. Iteration gives the product in (55.17). Since $P_m = \alpha_m P_{m-1}$, Theorem 55.2 gives

$$\sum_{m=1}^N \frac{|f_m|}{\alpha_m} = \sum_{m=1}^N P_{m-1} \frac{|f_m|}{P_m} \leq \frac{3\pi}{8} \sum_{m=1}^N \frac{|f_m|}{P_m} \leq \frac{\pi}{96}.$$

Apply $1 + t \leq e^t$ and $P_N \leq 3\pi/8$.

For an orthogonal Q_m , the actual split gives the exact identity

$$\|x_m\|_m^2 = \alpha_m^2 \|(I - Q_m)y_m\|_m^2 + (\alpha_m + f_m)^2 \|Q_m y_m\|_m^2. \quad (55.19)$$

At $m = 1$, $\alpha_1 = 192/175$ and $f_1 = 8/525$, so $\alpha_1 + f_1 = (73/72)\alpha_1$. For $m \geq 2$, $f_m < 0$ and

$$\alpha_m + f_m = \alpha_m(1 - \rho_{m-1}) + \rho_m > 0$$

because $0 \leq \rho_j \leq 8/525 < 1$. Thus the largest of the two squared coefficients in (55.19) is α_m^2 for every $m \geq 2$. The first step costs at most $(73/72)\alpha_1$, and all later steps cost at most α_m . Multiplication proves (55.18).

For sharpness take the fixed space $\mathbb{R}^2$, $\mathcal{U}_m = I$, $x_0 = (1, 0)$, Q_1 the projection onto the first axis, and every Q_m for $m \geq 2$ the projection onto the second axis. Then $x_N = (73/72)P_N x_0$, whose norms approach $73\pi/192$. $\square$

Corollary 55.7 (Derived packet budget with changing residual directions). In the orthogonal-projection case,

$$\sup_{m \geq 1} \|e_m\|_m \leq \frac{73\pi}{192} \|x_0\|_0.$$

For the general contractive readout, the same assertion holds with $(3\pi/8)e^{\pi/96}$. These are consequences of the inherited recurrence and its initial state.

Proof. $\|e_m\|_m \leq \|y_m\|_m \leq \|x_{m-1}\|_{m-1}$. The case $m = 1$ is bounded by $\|x_0\|_0$; every later case follows from Theorem 55.6. $\square$

The co-transported packet theorem transports a fixed initial direction. The present theorem permits a different residual readout at every event. In both constructions the remainder is drawn from the retained state. Its appearance in $f_m e_m$ is the specified coefficient update on that state, with no additional independently supplied packet.

56. Transported finite vortex and exact circulation readout

Let T_n be the finite transport map of the transverse carrier at stage n . Let B_n be a positive-flux coordinate map and assume

$$B_{n+1}T_n = \lambda_n O_n B_n, \quad O_n^T O_n = I, \quad \lambda_n > 0. \quad (56.1)$$

Set $G_n = B_n^T B_n$. Then

$$T_n^T G_{n+1} T_n = \lambda_n^2 G_n. \quad (56.2)$$

For a polygonal transverse carrier with area A_n and perimeter L_n ,

$$A_{n+1} = \lambda_n^2 A_n, \quad L_{n+1} = \lambda_n L_n, \quad \frac{L_{n+1}}{\sqrt{A_{n+1}}} = \frac{L_n}{\sqrt{A_n}}. \quad (56.3)$$

Proposition 56.1 (Transported complete-state norm). For a carrier coefficient define

$$\|a\|_n^2 = \frac{a^T G_n a}{A_n}.$$

Then the already specified geometry (56.2)–(56.3) gives

$$\|T_n a\|_{n+1}^2 = \frac{a^T T_n^T G_{n+1} T_n a}{A_{n+1}} = \frac{a^T G_n a}{A_n} = \|a\|_n^2. \quad (56.4)$$

The identity holds on the direct sum of all retained carrier copies. In particular the completed-aperture handoff $(T_n \oplus T_n)\mathcal{S}_n(c_n, r_n)$ preserves their complete normalized norm, when C_n is the orthogonal involution for G_n .

Proof. Insert $T_n^T G_{n+1} T_n = \lambda_n^2 G_n$ and $A_{n+1} = \lambda_n^2 A_n$; the two factors cancel exactly. Sum over the retained copies and apply (54.9) in the incoming metric. This uses the geometric area already assigned to the carrier; it introduces no change to the event dynamics. $\square$

Let the edge carrier coefficient satisfy

$$V_{n+1,e} = T_n V_{n,e} + K_{n,e}, \quad (56.5)$$

where $K_{n,e}$ is the wake injected or re-exposed at the event.

Definition 56.2 (Finite circulation and finite vortex). For an oriented polygonal carrier P_n , define the edge one-cochain

$$a_{n,e} := V_{n,e}^T G_n \ell_{n,e}, \quad (56.6)$$

the finite circulation

$$\Gamma_n^{\text{fin}} := \sum_{e \in \partial P_n} a_{n,e}, \quad (56.7)$$

and the finite vortex density

$$\beta_n^{\text{fin}} := \frac{\Gamma_n^{\text{fin}}}{A_n}. \quad (56.8)$$

The carrier is vortical at stage n when $\Gamma_n^{\text{fin}} \neq 0$.

Theorem 56.3 (Pure transport cancels against scale). The finite circulation-density increment is

$$\beta_{n+1}^{\text{fin}} - \beta_n^{\text{fin}} = \frac{1}{A_{n+1}} \sum_e K_{n,e}^T G_{n+1} T_n \ell_{n,e}. \quad (56.9)$$

Thus pure carrier transport contributes no additive normalized-circulation increment after geometric area rescaling; the displayed increment is the retained wake pairing.

Proof. Insert (56.5) and $\ell_{n+1,e} = T_n \ell_{n,e}$ into (56.8). The pure term is

$$\frac{1}{A_{n+1}} \sum_e V_{n,e}^T T_n^T G_{n+1} T_n \ell_{n,e} = \frac{\lambda_n^2}{A_{n+1}} \sum_e V_{n,e}^T G_n \ell_{n,e} = \beta_n^{\text{fin}}$$

by (56.2) and (56.3). The remaining term is (56.9).

Define $r_{n,e} := V_{n,e}/\sqrt{A_n}$, and let R_n be the least upper bound of $\|r_{n,e}\|_{G_n}$ over occupied edges. Cauchy–Schwarz and (56.3) give

$$|\beta_n^{\text{fin}}| \leq \frac{L_0}{\sqrt{A_0}} R_n. \quad (56.10)$$

For each occupied edge let $x_{n,e}$ be its complete retained carrier state, including its residual channels, and suppose $V_{n,e}$ is a canonical readout of norm at most one in the metric above. Put $S_n = \sup_e \|x_{n,e}\|_n$. Then $R_n \leq S_n$. For these states obeying the recurrence of Corollary 55.4 with uniform E_w , one obtains

$$|\beta_N^{\text{fin}}| \leq \frac{L_0}{\sqrt{A_0}} \left(\frac{3\pi}{8} S_0 + \frac{E_w \pi}{96} \right). \quad (56.11)$$

If every retained edge state follows the inherited orthogonal-projection recurrence, Theorem 55.6 instead gives

$$|\beta_N^{\text{fin}}| \leq \frac{L_0}{\sqrt{A_0}} \frac{73\pi}{192} S_0.$$

Here S_0 includes every initial remnant capable of later entering the circulation readout. Replacing it by the initially visible R_0 would omit precisely the ancestry that successive wakes retain.

Theorem 56.4 (Finite vortex ancestry). For (56.5),

$$V_N = T_{N-1} \cdots T_0 V_0 + \sum_{j=0}^{N-1} T_{N-1} \cdots T_{j+1} K_j, \quad (56.12)$$

where the empty product for $j = N - 1$ is the identity. Thus the stage- N finite vortex carrier is the transported ancestral carrier plus every earlier wake input funneled through all subsequent motion maps.

Proof. Induction on N . The case $N = 1$ is (56.5). Assuming the formula at N and applying $V_{N+1} = T_N V_N + K_N$ gives the formula at $N + 1$.

For a linear carrier readout $V_n = \mathcal{V}_n a_n$ along the occupied history, the native wake law gives the exact identity

$$K_n = (\mathcal{V}_{n+1} M_n - T_n \mathcal{V}_n) a_n. \quad (56.13)$$

Thus an additive increment in the circulation readout can be the re-exposure of content already present in a_n . Formula (56.12) remains an exact ancestry identity, but its terms are not thereby new independent inputs to the complete-state norm. The bound is applied to the complete wake first and then to its canonical readout.

The separatrix supplies the sign geometry, the wake supplies inherited loading, and motion composes the two. The word “vortex” is therefore already defined in the finite category by nonzero circulation; the continuum curl is not its primitive definition.

56.1. Exact observational circulation correspondence

Triangulate the polygonal carrier and let W_n denote the Whitney reconstruction of the edge one-cochain $a_n = (a_{n,e})_e$. Thus

$$u_n^b := W_n(a_n) \quad (56.14)$$

is an observational one-form satisfying the edge interpolation property

$$\int_e u_n^b = a_{n,e}. \quad (56.15)$$

Theorem 56.5 (Finite-vortex/continuum-readout correspondence). For every reconstructed carrier,

$$\Gamma_n^{\text{fin}} = \oint_{\partial P_n} u_n \cdot d\ell. \quad (56.16)$$

If $\omega_n := \nabla \times u_n$ denotes the curl of the observational reconstruction in the piecewise-smooth or weak sense, then

$$\beta_n^{\text{fin}} = \frac{1}{A_n} \int_{P_n} \omega_n \cdot n \, dA. \quad (56.17)$$

Proof. Sum (56.15) over the oriented boundary edges to obtain (56.16). Stokes’ theorem for the reconstructed one-form gives $\oint_{\partial P_n} u_n \cdot d\ell = \int_{P_n} \omega_n \cdot n \, dA$. Divide by A_n .

No limit $A_n \rightarrow 0$ is required. Classical vorticity is an observational readout of the finite circulation cochain at the chosen resolution.

57. Finite event ledger and exact continuum projected affine balance

The reconstruction and cochain conventions are those of Whitney forms, de Rham theory, and discrete exterior calculus [23–27].

The finite-to-shadow bridge is stated event first. No continuum velocity field and no external clock are assumed before the finite ledger is defined. The only continuous parameter introduced below is the affine observational coordinate induced by the completed finite event endpoints.

57.1. Wake-realization time and finite observational geometry

Let $\mathcal{H}_{\text{fin}}$ denote the set of occupied finite histories. The CFT ordering/wake structure supplies a strictly increasing wake-realization clock map

$$\Theta_w : \mathcal{H}_{\text{fin}} \longrightarrow \mathbb{R}, \quad X_{\leq k} \prec X_{\leq k+1} \implies \Theta_w(X_{\leq k+1}) > \Theta_w(X_{\leq k}). \quad (57.1)$$

Define

$$\tau_k := \Theta_w(X_{\leq k}), \quad \Delta\tau_k := \tau_{k+1} - \tau_k > 0. \quad (57.2)$$

Thus τ_k is not a freely supplied continuum time coordinate. It is the scalar readout of the occupied, wake-conditioned finite history. The symbol t will appear only after reconstruction, as the affine observation coordinate on the already determined interval $[\tau_k, \tau_{k+1}]$.

Theorem 57.1 (Nonaccumulation from the anticonformal event clock). Use the elementary clock law of [18], equations (12)–(14) and (25). To distinguish elementary durations from the endpoint values τ_k , write that law as

$$\sigma_n = \sigma_* + \mathcal{J}_\tau(Z_n, w_n, \Lambda_n), \quad |\mathcal{J}_\tau(Z_n, w_n, \Lambda_n)| \leq \eta\sigma_*, \quad (57.3)$$

$$s_{n+1} = s_n + \sigma_n, \quad \sigma_* = 0.5 \text{ ns}, \quad 0 \leq \eta < 1. \quad (57.4)$$

For any completed event subsequence $q_0 < q_1 < \dots$ of elementary indices, set $\tau_k = s_{q_k}$. Then, with $h_* = (1 - \eta)\sigma_* > 0$,

$$\tau_{k+1} - \tau_k \geq (q_{k+1} - q_k)h_* \geq h_*, \quad (57.5)$$

$$\tau_N \geq \tau_0 + Nh_*. \quad (57.6)$$

Consequently no infinite completed event sequence accumulates at a finite wake-clock value. For $T \geq \tau_0$, the number of completed advances by T satisfies

$$N(T) := \max\{k : \tau_k \leq T\} \leq \left\lfloor \frac{T - \tau_0}{h_*} \right\rfloor. \quad (57.7)$$

For two successive independent source admissions governed by the complete 1:3 opportunity word, the source's four-gate rule further gives a lag of at least $4h_*$. This stronger lag concerns independent source admissions, not every elementary advance or every completed flip.

Proof. Equation (57.3) gives $\sigma_n \geq h_*$ pointwise on the occupied history. Therefore

$$\tau_{k+1} - \tau_k = \sum_{n=q_k}^{q_{k+1}-1} \sigma_n \geq (q_{k+1} - q_k)h_*.$$

The integer difference is at least one. Summation proves (57.6) and hence (57.7). The complete-cell admission rule in Definition 2.1 and equation (210) of [18] makes the corresponding integer difference at least four. No independence, stationarity, or sign cancellation of the wake-dependent jitter is used. $\square$

Thus the source clock supplies the required nonaccumulation directly. The numerical value of σ_* fixes its physical calibration; the proof uses only its positivity and the single source bound $\eta < 1$. Strict monotonicity of an arbitrary scalar readout alone is a weaker statement. The endpoint clock used for this source realization is the specified s_{q_k} , before any separate laboratory synchronization.

Definition 57.2 (Operational synchronization with a laboratory clock). A laboratory clock is represented by a second finite occupied history $C_{\leq n}$ with its own strictly increasing wake-realization map

$$\Theta_w^{\text{ref}}(C_{\leq n}) =: \vartheta_n. \quad (57.8)$$

Let $\mathcal{C} \subset \mathbb{N} \times \mathbb{N}$ be the experimentally recorded coincidence relation, where $(k, n) \in \mathcal{C}$ means that fluid event k and reference-clock event n are registered as coincident. For any single-valued calibration branch $n = n(k)$ define

$$t_k^{\text{lab}} := \Theta_w^{\text{ref}}(C_{\leq n(k)}). \quad (57.9)$$

Thus laboratory time is a comparison between two finite wake-realized histories, not an ontologically independent parameter supplied to the fluid dynamics.

Proposition 57.3 (Calibration map). On every interval of events for which the coincidence relation is order-preserving, there is a monotone calibration map Ξ_{cal} on the observed event set such that

$$t_k^{\text{lab}} = \Xi_{\text{cal}}(\tau_k). \quad (57.10)$$

If the two clocks are experimentally calibrated at constant relative rate over that regime, then $\Xi_{\text{cal}}(\tau) = a\tau + b$ with $a > 0$.

Proof. Order preservation makes the assignment $\tau_k \mapsto t_k^{\text{lab}}$ single-valued and monotone on the observed event set. The affine formula is exactly the constant-rate calibration subclass. No continuum clock is used to generate either event sequence.

For the exact commuting theorem fix the periodic domain $\Omega = \mathbb{T}^3$. For each observational resolution M , let K_M be a finite periodic oriented tetrahedral complex on Ω . Let

$$C_M^r := C^r(K_M; \mathbb{R}), \quad d_r : C_M^r \longrightarrow C_M^{r+1}, \quad d_{r+1}d_r = 0, \quad (57.11)$$

be its real cochain complex. Let H_r be positive-definite discrete Hodge matrices and let

$$W_M^r : C_M^r \longrightarrow \Lambda_M^r \quad (57.12)$$

be compatible Whitney-type reconstruction maps satisfying

$$dW_M^r = W_M^{r+1} d_r. \quad (57.13)$$

For the compatible Whitney class let

$$I_M^r : \Lambda_M^r \longrightarrow C_M^r, \quad (I_M^r \alpha)(\sigma^r) := \int_{\sigma^r} \alpha, \quad (57.14)$$

be the finite de Rham degree-of-freedom map, and require the exact interpolation identity

$$I_M^r W_M^r = I_{C_M^r}. \quad (57.15)$$

Thus the Whitney reconstruction is faithful on the finite cochain degrees of freedom. For $r = 2$ let $\mathcal{V}$ denote the standard vector proxy $\alpha = \iota_u \text{vol}$. Hence

$$P_M^u(m) := \mathcal{V} W_M^2 m =: u_m, \quad m \in C_M^2. \quad (57.16)$$

Proposition 57.4 (Faithfulness of the momentum readout). The compatible momentum reconstruction

$$P_M^u : C_M^2 \longrightarrow V_M := P_M^u(C_M^2)$$

is injective. In particular, it is an isomorphism from C_M^2 onto its finite-dimensional image V_M ; hence every later restriction of this map, including the closed-flux sector, is injective.

Proof. If $P_M^u(m) = P_M^u(n)$, invertibility of the vector proxy gives $W_M^2 m = W_M^2 n$. Applying I_M^2 and using (57.15) gives $m = n$.

Define the primary finite momentum state space

$$X_M := \ker(d_2 : C_M^2 \rightarrow C_M^3). \quad (57.17)$$

Proposition 57.5 (Divergence-preserving reconstruction). For every $m \in X_M$,

$$\nabla \cdot P_M^u(m) = 0. \quad (57.18)$$

Proof. By (57.13), $dW_M^2 m = W_M^3 d_2 m = 0$. Under the vector proxy, $d(\iota_{u_m} \text{vol}) = (\nabla \cdot u_m) \text{vol}$. Thus $\nabla \cdot u_m = 0$. Periodicity removes boundary flux terms.

Define the discrete coderivatives by

$$\delta_{r+1} := H_r^{-1} d_r^T H_{r+1} : C_M^{r+1} \rightarrow C_M^r \quad (57.19)$$

and the discrete Hodge Laplacian on primal two-cochains

$$A_M := d_1 \delta_2 + \delta_3 d_2. \quad (57.20)$$

It is positive semidefinite in the H_2 inner product. Our continuum-proxy sign convention is

$$-\Delta_M P_M^u(m) := \mathcal{V} W_M^2 A_M m. \quad (57.21)$$

Thus Δ_M is negative semidefinite, and $+\nu_M A_M$ in the finite ledger reconstructs as $-\nu_M \Delta_M$ on the left-hand side.

The pressure-gradient cochain operator is

$$G_M := -\delta_3, \quad \langle G_M \pi, v \rangle_{H_2} = -\langle \pi, d_2 v \rangle_{H_3}. \quad (57.22)$$

Pressure is therefore the multiplier dual to the discrete closed-flux constraint, not an independent primitive force.

57.2. Resolved quadratic flux and retained wake stress in the finite ledger

Let Q_M be a finite tensor-flux cochain space and

$$W_M^Q : Q_M \longrightarrow Z_M \subset L^2(\Omega; \mathbb{R}^{3 \times 3}) \quad (57.23)$$

be an injective reconstruction and hence an isomorphism onto its finite-dimensional image $Z_M = \text{Ran}(W_M^Q)$. The compatible quadratic model class is required to satisfy the explicit range condition

$$P_M^u(C_M^2) \otimes P_M^u(C_M^2) \subseteq \text{Ran}(W_M^Q) = Z_M. \quad (57.24)$$

Because W_M^Q is injective, condition (57.24) gives existence and uniqueness of the tensor-flux cochain representing every resolved quadratic product. Define that unique finite bilinear quadratic-product map

$$B_M : C_M^2 \times C_M^2 \longrightarrow Q_M \quad (57.25)$$

by the reconstruction identity

$$W_M^Q B_M(m, n) = P_M^u(m) \otimes P_M^u(n). \quad (57.26)$$

Equivalently, $B_M = (W_M^Q)^{-1} \circ (P_M^u \otimes P_M^u)$ on the stated finite image.

For resolution M , let W_M denote a finite-dimensional real vector space of wake coordinates containing the admissible vortex/separatrix carrier states at that observational resolution. The wake handoff and deposition operators below are linear on this wake leg. Let

$$a_k^w \in W_M, \quad a_{k+1}^w = M_k^w a_k^w, \quad M_k^w := K_{k+1,k} P_{L,k+1} B_k, \quad (57.27)$$

be the same source-to-successor handoff already proved in Section 52, now restricted to the fluid observational resolution.

The carrier geometry is itself event dependent. At event k write

$$\mathfrak{g}_k := (E_k, G_k, \{\ell_{k,e}\}_{e \in E_k}, A_k, \{\Xi_{M,k,e}\}_{e \in E_k}, \{V_{k,e}\}_{e \in E_k}),$$

where E_k is the occupied edge set, G_k the carrier metric, $\ell_{k,e}$ the oriented edge vectors, $A_k > 0$ the carrier area, $\Xi_{M,k,e} \in Q_M$ the tensor-flux deposition cochains, and $V_{k,e}$ the wake-coordinate extraction maps. Let

$$V_{k,e} : W_M \longrightarrow \mathbb{R}^{d_e}$$

be the linear coordinate functional extracting the coefficient carried by edge e , so that $V_{k,e}(a_k^w) = V_{k,e}$. The event-dependent wake-deposition operator is

$$D_{M,k} : W_M \longrightarrow Q_M, \quad D_{M,k}(a) := \sum_{e \in E_k} \frac{V_{k,e}(a)^T G_k \ell_{k,e}}{A_k} \Xi_{M,k,e}. \quad (57.28)$$

For fixed occupied event geometry $\mathfrak{g}_k$, $D_{M,k}$ is linear in the wake state. Its k -dependence records the evolving carrier rather than hiding that geometry inside a nominally fixed operator.

Define

$$r_k := D_{M,k}(a_k^w), \quad r_{k+1} := D_{M,k+1}(M_k^w a_k^w). \quad (57.29)$$

Hence r_k is no longer an externally supplied event datum in this model class; it is the deposition of the incoming finite wake on the carrier actually occupied at event k . The primitive external packet contains only genuinely new impulse data and the internally generated event interval.

Because the deposition map changes with the occupied carrier, exact closed evolution of the projected wake is a morphism between successive observed wake spaces, not an invariance statement for a single fixed kernel.

Lemma 57.6 (Event-dependent wake-deposition quotient criterion). Assume $D_{M,k}$, $D_{M,k+1}$, and M_k^w are linear on the wake leg. There exists a unique linear operator

$$R_{M,k}^w : \text{Ran}(D_{M,k}) \longrightarrow \text{Ran}(D_{M,k+1}) \quad (57.30)$$

satisfying

$$D_{M,k+1} M_k^w = R_{M,k}^w D_{M,k} \quad (57.31)$$

if and only if

$$M_k^w(\ker D_{M,k}) \subseteq \ker D_{M,k+1}. \quad (57.32)$$

When this holds,

$$r_{k+1} = R_{M,k}^w r_k. \quad (57.33)$$

Proof. If (57.32) holds, define $R_{M,k}^w(D_{M,k}a) := D_{M,k+1}M_k^w a$. If $D_{M,k}a = D_{M,k}b$, then $a - b \in \ker D_{M,k}$, so $M_k^w(a - b) \in \ker D_{M,k+1}$ and the definition is independent of representative. The induced map is unique on $\text{Ran}(D_{M,k})$. Conversely, (57.31) sends every $a \in \ker D_{M,k}$ to $D_{M,k+1}M_k^w a = 0$, proving (57.32).

Thus non-injectivity is compatible with closed observed wake dynamics precisely when the hidden fiber at event k is transported into the hidden fiber of the successor deposition map. The quotient itself is therefore event dependent, matching the evolving separatrix/vortex carrier.

The complete second-moment ledger is defined before reconstruction by

$$q_k = B_M(m_k, m_k) + r_k. \quad (57.34)$$

Thus the wake stress is a finite ledger component generated by propagated carrier ancestry, not a remainder manufactured by continuum projection.

For each interior or periodically identified face $F = K^+ \cap K^-$ choose n_F pointing from K^+ into K^- . Put $[v]_F = v^+ - v^-$ and $\{Q\}_F = (Q^+ + Q^-)/2$. Let ∇_h mean the elementwise gradient. Define the tensor-divergence cochain operator by the finite face-flux Riesz pairing

$$\begin{aligned} \langle C_M q, v \rangle_{H_2} = & - \sum_{K \in K_M} \int_K W_M^Q q : \nabla_h(P_M^u v) \, dx \\ & + \sum_{F \in \mathcal{F}_M} \int_F (\{W_M^Q\}_F n_F) \cdot [P_M^u v]_F \, dS. \end{aligned} \quad (57.35)$$

All traces in this formula are polynomial traces from their respective cells. Reversing the chosen face orientation reverses both n_F and the jump, leaving the pairing unchanged. Positive definiteness of H_2 gives one unique finite cochain $C_M q$. The periodic domain has no external boundary; its internal and periodically identified faces remain in the sum. Section 57.8 constructs the spaces explicitly and proves that this operator has nonzero closed-sector quadratic transport.

57.3. Finite ledger space, derived pressure, and the one-event law

Let

$$C_{M,0}^3 := \{\pi \in C_M^3 : \langle \pi, 1_M \rangle_{H_3} = 0\}, \quad (57.36)$$

where 1_M is the harmonic constant top cochain. The primitive event packet is

$$\lambda_k = (j_k, \Delta\tau_k) \in C_M^2 \times (0, \infty), \quad (57.37)$$

with $\Delta\tau_k$ fixed by the wake-realization map (57.2). Define

$$g_k := \frac{j_k}{\Delta\tau_k}. \quad (57.38)$$

Define the augmented primary finite state

$$Y_k := (m_k, a_k^w) \in Y_M := X_M \times W_M. \quad (57.39)$$

The completed finite event ledger is

$$L_k = (Y_k, r_k, q_k, j_k, \pi_k, Y_{k+1}, \Delta\tau_k), \quad (57.40)$$

where $r_k = D_{M,k}a_k^w$, $a_{k+1}^w = M_k^w a_k^w$, $r_{k+1} = D_{M,k+1}a_{k+1}^w$, and $m_k, m_{k+1} \in X_M$. The ledger state space is therefore the constrained subset

$$\mathcal{L}_M \subset Y_M \times Q_M \times Q_M \times C_M^2 \times C_{M,0}^3 \times Y_M \times (0, \infty), \quad (57.41)$$

with membership constrained by (57.29), (57.34), (57.37), (57.44), and (57.45).

Define the top-degree pressure Laplacian

$$L_{p,M} := d_2\delta_3 : C_{M,0}^3 \rightarrow C_{M,0}^3. \quad (57.42)$$

On the connected periodic three-complex, its only excluded harmonic mode is 1_M . For every $y \in C_M^2$,

$$\langle d_2y, 1_M \rangle_{H_3} = \langle y, \delta_3 1_M \rangle_{H_2} = 0, \quad (57.43)$$

so d_2y has the required mean-zero compatibility. The remaining uniqueness property follows from the energy identity below.

Lemma 57.7 (Mean-zero pressure invertibility). For the connected periodic complex with positive-definite Hodge inner products,

$$\ker(L_{p,M}) \cap C_{M,0}^3 = \{0\}.$$

Proof. If $L_{p,M}\pi = d_2\delta_3\pi = 0$, then

$$\langle \pi, d_2\delta_3\pi \rangle_{H_3} = \langle \delta_3\pi, \delta_3\pi \rangle_{H_2} = \|\delta_3\pi\|_{H_2}^2.$$

Hence $\delta_3\pi = 0$. On a connected periodic complex the degree-three harmonic sector consists only of constants, so $\pi = c1_M$. The mean-zero gauge implies $c = 0$, and therefore $\pi = 0$.

Hence the pressure cochain is uniquely determined in $C_{M,0}^3$ by

$$L_{p,M}\pi_k = d_2(C_M q_k - g_k). \quad (57.44)$$

Theorem 57.8 (Exact augmented one-event finite law with exogenous impulse packet).

For $Y_k = (m_k, a_k^w) \in Y_M$ and an admissible event packet (57.37), define r_k by (57.29), q_k by (57.34), g_k by (57.38), and π_k by (57.44). The one-event update is

$$\frac{m_{k+1} - m_k}{\Delta\tau_k} + C_M q_k + G_M \pi_k + \nu_M A_M m_k = g_k, \quad (57.45)$$

$$a_{k+1}^w = M_k^w a_k^w. \quad (57.46)$$

Then $m_{k+1} \in X_M$, so

$$E_{\text{fin}, M}^{k, \lambda_k} : Y_M \rightarrow Y_M, \quad Y_k \mapsto Y_{k+1} := (m_{k+1}, a_{k+1}^w), \quad (57.47)$$

is a well-defined one-event evolution on the augmented finite momentum–wake state.

Proof. Apply d_2 to (57.45). The finite Hodge identity $d\Delta = \Delta d$ gives $d_2 A_M m_k = 0$ because $d_2 m_k = 0$. Since $G_M = -\delta_3$,

$$\frac{d_2 m_{k+1}}{\Delta\tau_k} + d_2 C_M q_k - d_2 \delta_3 \pi_k = d_2 g_k.$$

Equation (57.44) makes the last three terms cancel, so $d_2 m_{k+1} = 0$.

Theorem 57.9 (Unconditional finite continuation of the native event). On the periodic complex constructed in Theorem 57.19, every initial $(m_0, a_0^w) \in X_M \times W_M$ and every admissible finite sequence of the event data in (57.27)–(57.45) determine a unique finite history through its last event. No additional contraction, deposition-quotient, or triadic-cover hypothesis is needed for this conclusion.

In particular, define the fixed constraint projection

$$P_M = I - \delta_3 L_{p, M}^{-1} d_2. \quad (57.48)$$

The original event, with $\delta_k = \Delta\tau_k > 0$, is exactly

$$m_{k+1} = m_k + P_M j_k - \delta_k [P_M C_M \{B_M(m_k, m_k) + D_{M, k} a_k^w\} + \nu_M A_M m_k], \quad (57.49)$$

$$a_{k+1}^w = M_k^w a_k^w. \quad (57.50)$$

This is pressure elimination in the stated law, with all its wake deposition retained; it is not a different event rule.

Proof. The inverse in (57.48) acts on the mean-zero top-cochain space. Equation (57.43) puts every $d_2 y$ in that space, and Lemma 57.7 proves invertibility there. Thus $d_2 P_M = 0$, while $P_M m = m$ for $m \in X_M$. The correction $y - P_M y$ lies in $\text{Ran } \delta_3$, which is H_2 -orthogonal to X_M ; hence P_M is its orthogonal projection. Also

$$d_2 A_M m = d_2 d_1 \delta_2 m + d_2 \delta_3 d_2 m = 0 \quad (m \in X_M).$$

Substitute $\pi_k = L_{p,M}^{-1} d_2(C_M q_k - j_k / \delta_k)$ into (57.45); rearrangement gives (57.49). All terms are finite matrix operations or the specified finite quadratic map, and δ_k is positive. The successor momentum is closed because every term on its right lies in X_M . The linear wake map supplies its unique successor, whose entire outgoing wake vector is retained in the state. Induction over the finite event sequence gives existence and uniqueness at every finite depth. $\square$

Lemma 57.10 (Finite-data bound for the actual nonlinear event). Choose fixed norms on the finite cochain and wake spaces. Put

$$b_M = \sup_{\|v\|=\|w\|=1} \|B_M(v, w)\|, \quad c_M = \|P_M C_M\|, \quad S_k = 1 + \|m_k\| + \|a_k^w\|.$$

All these operator norms are finite. For the actual occupied event data define

$$C_k = \max\{1 + \|P_M j_k\|, 1 + \delta_k |\nu_M| \|A_M\|, \delta_k c_M \|D_{M,k}\| + \|M_k^w\|, \delta_k c_M b_M\}. \quad (57.51)$$

Then $C_k \geq 1$ and

$$S_N \leq S_0^{2^N} \prod_{j=0}^{N-1} C_j^{2^{N-1-j}} < \infty \quad (N < \infty). \quad (57.52)$$

No coefficient is required to be bounded uniformly in k or M .

Proof. Apply the triangle inequality and the displayed operator norms to the unchanged update (57.49)–(57.50). With $x = \|m_k\|$ and $y = \|a_k^w\|$ this gives

$$\begin{aligned} S_{k+1} &\leq 1 + \|P_M j_k\| + (1 + \delta_k |\nu_M| \|A_M\|)x \\ &\quad + (\delta_k c_M \|D_{M,k}\| + \|M_k^w\|)y + \delta_k c_M b_M x^2 \\ &\leq C_k(1 + x + y + x^2) \leq C_k(1 + x + y)^2 = C_k S_k^2. \end{aligned}$$

Finite induction gives (57.52); the exponent of each earlier C_j doubles at every subsequent step. All coefficients are evaluated on the same occupied history as the original event. $\square$

Theorem 57.11 (Global continuation in the source wake clock). Fix a finite observational resolution M , any initial $(m_0, a_0^w) \in X_M \times W_M$, and an infinite admissible sequence of the specified native event data, clocked by (57.3)–(57.4) with one fixed $\eta < 1$. The actual law (57.49)–(57.50) determines a unique finite state at every event and continues through every finite wake-clock interval $[\tau_0, T]$. On each such interval its states, wake depositions, pressures, and affine velocity readout are bounded.

More precisely, for the already specified reconstruction (57.73),

$$\sup_{\tau_0 \leq t \leq T} \|u_M(t)\| \leq \max_{0 \leq k \leq N(T)+1} \|P_M^u m_k\| < \infty, \quad (57.53)$$

$$\text{Lip}_{[\tau_0, T]}(u_M) \leq \frac{1}{h_*} \max_{0 \leq k \leq N(T)} \|P_M^u(m_{k+1} - m_k)\| < \infty. \quad (57.54)$$

Here any fixed norm on the finite-dimensional velocity space may be used. The exact projected balance (57.75) holds almost everywhere in clock time, and the readout is divergence free throughout. These conclusions require no contraction of the occupied wake map, no deposition quotient, and no bound uniform over all event indices.

Proof. Theorem 57.9 gives a unique finite solution on each finite prefix. Uniqueness makes these prefixes agree on their overlaps, giving the state at every nonnegative event index. By Theorem 57.1, $\tau_k \rightarrow \infty$ and at most $N(T)$ advances occur by T . The next endpoint is also finite. Thus the entire interval $[\tau_0, T]$ uses only the finitely many ledgers through event $N(T)$. Every state, deposition, pressure, and event coefficient on those ledgers is finite by the original event formulas, so their finitely many norms have a finite maximum. For an explicit bound from those finite data, set $n_T = N(T) + 1$ and $\overline{C}_T = \max_{0 \leq j < n_T} C_j$. Since $\overline{C}_T, S_0 \geq 1$, Lemma 57.10 gives

$$\max_{0 \leq k \leq n_T} S_k \leq \overline{C}_T^{2^{n_T}-1} S_0^{2^{n_T}} < \infty.$$

On each event interval the observed velocity is the affine combination of its two finite endpoint velocities. Convexity of the norm gives (57.53). Its slope is $P_M^u(m_{k+1} - m_k)/(\tau_{k+1} - \tau_k)$, whose norm is bounded as in (57.54). The endpoint values agree between successive intervals. Splitting any time interval at its finitely many event endpoints proves the stated Lipschitz bound. The reconstructed one-event identity and closed-cochain constraint give the last claims. $\square$

For each fixed resolution M , the theorem gives the complete native history through every finite source-clock interval, with its inherited wake retained. Its bounds depend on M , the interval endpoint T , the initial state, and the occupied source history. The proof composes finitely many exact events on each such interval; it uses neither $M \rightarrow \infty$ nor an event-duration limit.

Remark 57.12 (Driven versus autonomous event dynamics). The map $E_{\text{fin},M}^{k,\lambda_k}$ is an exact wake-closed one-event map, but it is not called autonomous in the dynamical-systems sense when j_k , $\Delta\tau_k$, or M_k^w are supplied by an event-indexed or history-dependent source packet. The exact periodic commutation theorem below does not require autonomy.

Proposition 57.13 (Autonomous closure on an enlarged finite source state). Let $\mathcal{Z}_M$ be a finite source/controller state space and let $\mathfrak{G}_M$ be the class of admissible occupied carrier/deposition geometries at resolution M . An element of $\mathfrak{G}_M$ has the form

$$\mathfrak{g} = (E, G, \{\ell_e\}_{e \in E}, A, \{\Xi_{M,e}\}_{e \in E}, \{V_e\}_{e \in E}), \quad A > 0, \quad (57.55)$$

where the final family consists of linear maps $V_e : W_M \rightarrow \mathbb{R}^{d_e}$. Suppose there are event-label-independent maps

$$J_M : \mathcal{Z}_M \rightarrow C_M^2, \quad \Delta\Theta_{w,M} : \mathcal{Z}_M \rightarrow (0, \infty), \quad \mathcal{M}_M : \mathcal{Z}_M \rightarrow \text{End}(W_M), \quad \mathcal{G}_M : \mathcal{Z}_M \rightarrow \mathfrak{G}_M, \quad (57.56)$$

and

$$F_M : \mathcal{Z}_M \times X_M \times W_M \rightarrow \mathcal{Z}_M. \quad (57.57)$$

For every geometry $\mathfrak{g} \in \mathfrak{G}_M$, let

$$D_M(\mathfrak{g}) \in \text{Hom}(W_M, Q_M) \quad (57.58)$$

be the deposition operator obtained from the edge formula (57.28). Assume the controller generates the entire occupied geometry and all event data by

$$j_k = J_M(s_k), \quad \Delta\tau_k = \Delta\Theta_{w,M}(s_k), \quad M_k^w = \mathcal{M}_M(s_k), \quad (57.59)$$

$$\mathfrak{g}_k = \mathcal{G}_M(s_k), \quad D_{M,k} = D_M(\mathfrak{g}_k), \quad s_{k+1} = F_M(s_k, m_k, a_k^w). \quad (57.60)$$

In particular,

$$E_k = E(\mathfrak{g}_k), \quad G_k = G(\mathfrak{g}_k), \quad A_k = A(\mathfrak{g}_k), \quad \ell_{k,e} = \ell_e(\mathfrak{g}_k), \quad \Xi_{M,k,e} = \Xi_{M,e}(\mathfrak{g}_k), \quad V_{k,e} = V_e(\mathfrak{g}_k) \quad (57.61)$$

Then on

$$\hat{Y}_M := X_M \times W_M \times \mathcal{Z}_M$$

the augmented finite dynamics is generated by a single event-label-independent map

$$\hat{E}_{\text{fin},M} : \hat{Y}_M \rightarrow \hat{Y}_M. \quad (57.62)$$

Proof. Given (m, a^w, s) , equations (57.59)–(57.60) determine the impulse j , the wake-realized interval $\Delta\tau > 0$, the wake handoff M^w , and the entire occupied carrier geometry $\mathfrak{g} = \mathcal{G}_M(s)$, including its edge set, metric, area, oriented edge vectors, tensor-flux deposition cochains, and wake-coordinate extraction maps. Hence the deposition operator is not hidden in an external event label: it is the derived map $D_M(s) := D_M(\mathcal{G}_M(s))$. The deposited wake $r = D_M(s)a^w$, quadratic flux $q = B_M(m, m) + r$, gauge-fixed pressure, momentum update, wake update, and controller update are therefore functions of (m, a^w, s) alone. Their composition defines (57.62) without explicit dependence on k or on externally supplied carrier geometry.

Remark 57.14 (Driven class versus geometry-closed autonomous subclass). For the general driven event theorem, the sequence $\mathfrak{g}_k$ may be supplied as event data; no autonomy claim is made. Proposition 57.13 is stronger: the state s_k determines $\mathfrak{g}_k$ itself, so every object entering $D_{M,k}$ is generated by the enlarged finite state. This is the precise sense in which the autonomous subclass is geometry closed.

57.4. Exact endpoint projection before any continuum interpolation

Define the endpoint velocity states

$$u_M^k := P_M^u(m_k), \quad u_M^{k+1} := P_M^u(m_{k+1}), \quad (57.63)$$

and the finite-ledger readouts

$$Q_M^k := W_M^Q q_k, \quad \tau_M^k := W_M^Q r_k, \quad p_M^k := S_M^3 W_M^3 \pi_k, \quad f_M^k := P_M^u(g_k). \quad (57.64)$$

By (57.26) and (57.34),

$$Q_M^k = u_M^k \otimes u_M^k + \tau_M^k. \quad (57.65)$$

Define the induced observational operators

$$\text{Div}_M Q_M^k := P_M^u(C_M q_k), \quad (57.66)$$

$$\text{Grad}_M p_M^k := P_M^u(G_M \pi_k), \quad (57.67)$$

$$-\Delta_M u_M^k := P_M^u(A_M m_k). \quad (57.68)$$

Let

$$N_M(u_M^k) := \text{Div}_M(u_M^k \otimes u_M^k) = P_M^u C_M B_M(m_k, m_k). \quad (57.69)$$

Theorem 57.15 (Exact projected one-event identity). Every completed event ledger satisfies the discrete-time observational identity

$$\frac{u_M^{k+1} - u_M^k}{\Delta \tau_k} + \text{Div}_M Q_M^k + \text{Grad}_M p_M^k - \nu_M \Delta_M u_M^k - f_M^k = 0. \quad (57.70)$$

Equivalently, using the finite split (57.65),

$$\frac{u_M^{k+1} - u_M^k}{\Delta \tau_k} + N_M(u_M^k) = -\text{Grad}_M p_M^k + \nu_M \Delta_M u_M^k + f_M^k - \text{Div}_M \tau_M^k. \quad (57.71)$$

Both endpoint velocities are exactly divergence free.

Proof. Apply the linear map P_M^u to (57.45) and use (57.66)–(57.68). The wake form follows from (57.65). Divergence freedom follows from Proposition 57.5 and Theorem 57.8.

57.5. Continuum projected affine observation of the event identity

Only now introduce a continuous symbol. For the already determined wake-realization endpoints τ_k and τ_{k+1} , let

$$\theta(t) := \frac{t - \tau_k}{\Delta \tau_k}, \quad t \in [\tau_k, \tau_{k+1}], \quad (57.72)$$

and define the continuum projected affine interpolant

$$u_M(t) := (1 - \theta(t))u_M^k + \theta(t)u_M^{k+1}. \quad (57.73)$$

The variable t is therefore not a loose or external parameter: its event endpoints are the wake-realization values (57.2), and between those endpoints it is only the affine observational coordinate used to display the completed finite event.

For almost every $t \in (\tau_k, \tau_{k+1})$,

$$\partial_t u_M(t) = \frac{u_M^{k+1} - u_M^k}{\Delta \tau_k}. \quad (57.74)$$

Hence Theorem 57.15 is equivalently displayed as

$$\partial_t u_M(t) + \text{Div}_M Q_M^k = -\text{Grad}_M p_M^k + \nu_M \Delta_M u_M^k + f_M^k, \quad (57.75)$$

with the event quantities on the right fixed by the finite ledger at the left endpoint. In wake-split form,

$$\partial_t u_M(t) + N_M(u_M^k) = -\text{Grad}_M p_M^k + \nu_M \Delta_M u_M^k + f_M^k - \text{Div}_M \tau_M^k. \quad (57.76)$$

Moreover,

$$\nabla \cdot u_M(t) = 0 \quad (57.77)$$

for every t on the event interval because $u_M(t)$ is a convex combination of two closed-cochain reconstructions.

Equation (57.75) records the finite endpoint balance by affine interpolation. Its flux, pressure, viscosity, and forcing are fixed by the event's left-endpoint ledger. A smooth observational representation with instantaneous nonlinear terms is specified in Section 60.

57.6. Map domains and the compatible commuting diagram

Define the continuum projected affine state space

$$\mathcal{S}_M := \{(u, Q, p, f, \tau) : u \in V_M, \nabla \cdot u = 0, Q, \tau \in Z_M, p \in P_M, f \in V_M, Q = u \otimes u + \tau\}, \quad (57.78)$$

where

$$P_M := S_M^3 W_M^3 C_{M,0}^3.$$

The completed-event reconstruction

$$P_M^{\text{evt}} : \mathcal{L}_M \longrightarrow \mathcal{S}_M \times P_M^u(X_M) \quad (57.79)$$

returns $(u_M^k, Q_M^k, p_M^k, f_M^k, \tau_M^k)$ together with the endpoint u_M^{k+1} determined by the ledger.

Define the observed augmented state

$$\Pi_{M,k}(Y_k) := (u_M^k, \tau_M^k) := (P_M^u m_k, W_M^Q D_{M,k} a_k^w) \in V_M^{\text{div}} \times T_{M,k}^w, \quad T_{M,k}^w := W_M^Q \text{Ran}(D_{M,k}). \quad (57.80)$$

By Proposition 57.4, the inverse $(P_M^u)^{-1}$ is well defined on V_M . Likewise W_M^Q is invertible on its image. Hence an observed pair $(u, \tau) \in V_M^{\text{div}} \times T_{M,k}^w$ determines the finite momentum cochain $m = (P_M^u)^{-1}u$ and deposited wake cochain $r = (W_M^Q)^{-1}\tau$, while hidden wake ancestry inside the event- k fiber of $D_{M,k}$ need not be recovered.

For fixed external packet $\lambda_k = (j_k, \Delta\tau_k)$, define the continuum projected-affine velocity update

$$E_{\text{proj-aff},M}^{k,\lambda_k} : V_M^{\text{div}} \times T_{M,k}^w \rightarrow V_M^{\text{div}} \quad (57.81)$$

by (57.70), with $Q_M^k = u_M^k \otimes u_M^k + \tau_M^k$ and with pressure obtained from the unique finite representatives just described. This operator is not independently postulated; it is the reconstructed momentum component of the augmented finite event law.

Theorem 57.16 (Exact compatible one-event commutation). For every admissible finite depth, every observational resolution M , every admissible external event packet λ_k , and every $Y_k \in Y_M$,

$$P_M^u \pi_m E_{\text{fin},M}^{k,\lambda_k}(Y_k) = E_{\text{proj-aff},M}^{k,\lambda_k}(\Pi_{M,k}(Y_k)), \quad (57.82)$$

where $\pi_m(m, a) = m$. Consequently the compatible momentum commutation defect is identically zero:

$$R_{M,k} = 0. \quad (57.83)$$

If, in addition, the kernel-morphism criterion (57.32) holds, then $r_{k+1} = R_{M,k}^w r_k$ and the observed pair has a closed one-event morphism from $V_M^{\text{div}} \times T_{M,k}^w$ to $V_M^{\text{div}} \times T_{M,k+1}^w$: faithfulness of P_M^u removes hidden momentum ambiguity, while the event-dependent wake quotient removes only the unobserved ancestry carried from $\ker D_{M,k}$ into $\ker D_{M,k+1}$.

Proof. The upper route advances the augmented finite state by (57.45) and (57.46), then reconstructs its outgoing momentum component. The lower route reconstructs the incoming momentum and wake stress and applies the endpoint law (57.70). The tensor product term commutes by (57.26); tensor divergence, pressure gradient, viscosity, and forcing commute by definitions (57.66)–(57.68) and $f_M^k = P_M^u(g_k)$; incompressibility commutes by (57.18). Thus the endpoint velocities coincide term by term. For the final statement, equality of observed velocities implies equality of the finite momentum cochains by Proposition 57.4; therefore $B_M(m, m)$, $A_M m$, and the derived pressure response agree for equal observed (u, τ) and the same event packet. Equality of deposited wake stresses propagates by Lemma 57.6, and injectivity of W_M^Q identifies equal reconstructed wake stresses with equal deposited cochains. Hence the observed pair has a well-defined one-event morphism between the successive observed wake spaces.

Corollary 57.17 (Exact complete-wake observation without a quotient condition). Keep the retained wake already present in Y_M and define

$$\widehat{\Pi}_M(m, a) = (P_M^u m, a) \quad \text{on } X_M \times W_M.$$

This is an isomorphism onto $V_M^{\text{div}} \times W_M$, where $V_M^{\text{div}} = P_M^u(X_M)$. For each specified native event there is an exact observed update

$$\widehat{E}_{\text{obs}, M}^{k, \lambda_k} = \widehat{\Pi}_M E_{\text{fin}, M}^{k, \lambda_k} \widehat{\Pi}_M^{-1}, \quad (57.84)$$

and this commutation holds through every finite succession of events. The kernel-morphism condition (57.32) is unnecessary for this complete-wake observation.

Proof. Faithfulness of P_M^u gives the inverse $(u, a) \mapsto ((P_M^u)^{-1}u, a)$. Equations (57.49)–(57.50) then compute the unique outgoing pair. Multiplying the commuting identities for successive events cancels each adjacent inverse and reconstruction. If the source and geometry are generated by the controller of Proposition 57.13, include that same controller state on both sides; the proof is unchanged. $\square$

Only the narrower observation $(u, W_M^Q D_{M,k} a)$ asks whether the present deposited stress alone determines its successor. Its fiber condition must not be imposed on the native full-wake evolution. A remnant hidden from the current deposition may remain in a and contribute at a later event, exactly as in Theorem 52.4. The complete observation above keeps those coordinates and does not add a new physical state or alter a single handoff. It makes no claim that the thinner deposition observation is injective or automatically closed. For an arbitrary noncompatible observational reconstruction $\widetilde{P}_M$, define the measurable one-event defect by

$$\widetilde{R}_{M,k} := \frac{\widetilde{u}_M^{k+1} - \widetilde{u}_M^k}{\Delta \tau_k} + \widetilde{\text{Div}}_M \widetilde{Q}_M^k + \widetilde{\text{Grad}}_M \widetilde{p}_M^k - \nu_M \widetilde{\Delta}_M \widetilde{u}_M^k - \widetilde{f}_M^k. \quad (57.85)$$

No vanishing is asserted outside the compatible class.

For a linear noncompatible readout, the defect can nevertheless be bounded rather than left purely qualitative. Define the operator commutators

$$C_{C,M} := \widetilde{P}_M^u C_M - \widetilde{\text{Div}}_M \widetilde{W}_M^Q, \quad (57.86)$$

$$C_{G,M} := \widetilde{P}_M^u G_M - \widetilde{\text{Grad}}_M \widetilde{P}_M^p, \quad (57.87)$$

$$C_{A,M} := \widetilde{P}_M^u A_M + \widetilde{\Delta}_M \widetilde{P}_M^u, \quad (57.88)$$

$$C_{F,M}(g) := \widetilde{P}_M^u g - \widetilde{f}_M(g). \quad (57.89)$$

Because the time increment of a linear readout commutes with finite differencing, applying $\widetilde{P}_M^u$ to the one-event law and subtracting the reconstructed balance gives

$$\widetilde{R}_{M,k} = -C_{C,M} q_k - C_{G,M} \pi_k - \nu_M C_{A,M} m_k + C_{F,M}(g_k). \quad (57.90)$$

Hence, in any chosen compatible norms,

$$\|\tilde{R}_{M,k}\| \leq \|C_{C,M}\| \|q_k\| + \|C_{G,M}\| \|\pi_k\| + \nu_M \|C_{A,M}\| \|m_k\| + \|C_{F,M}(g_k)\|. \quad (57.91)$$

The compatible Whitney class is precisely the zero-commutator case.

Proposition 57.18 (Eventwise observational quotient morphism). Assume the wake kernel-morphism criterion (57.32). Define the event- k observational relation

$$Y \sim_{M,k} Z \iff \Pi_{M,k}(Y) = \Pi_{M,k}(Z).$$

Then

$$Y \sim_{M,k} Z \implies E_{\text{fin},M}^{k,\lambda_k}(Y) \sim_{M,k+1} E_{\text{fin},M}^{k,\lambda_k}(Z).$$

Consequently the finite event map induces a well-defined morphism

$$E_{M,k} : Y_M / \sim_{M,k} \longrightarrow Y_M / \sim_{M,k+1},$$

even though the wake observation is generally non-injective.

Proof. If $\Pi_{M,k}(Y) = \Pi_{M,k}(Z)$, equality of the velocity components and Proposition 57.4 give equality of their finite momentum cochains. Equality of the reconstructed wake stresses and injectivity of W_M^Q give equality of their event- k deposited wake cochains. For the same external event packet, the momentum update therefore has identical B_M , wake, Hodge, and derived-pressure inputs, so Theorem 57.16 gives equal outgoing observed velocities. Lemma 57.6 gives equal successor deposited wakes under $D_{M,k+1}$. Hence the two outgoing states are equivalent under $\sim_{M,k+1}$, and the induced quotient morphism is well defined.

The exact direction of explanation is therefore

$$\begin{aligned} \text{finite primary event dynamics} &\longrightarrow \text{compatible Whitney projection} \\ &\longrightarrow \text{exact divergence-free} \\ &\text{continuum projected affine balance.} \end{aligned} \quad (57.92)$$

The projection is generally non-injective. Exact preservation of the projected balance therefore does not imply recovery of the full finite sign carrier, wake ancestry, or refinement history.

57.7. Finite wake-quotient constructions and reported checks

An explicit finite example records how different inherited wakes can have the same deposited observation through every successor. Let

$$D(x, y, z) = (x, y), \quad M = \text{diag}(1/2, 1/3, 7/10), \quad R = \text{diag}(1/2, 1/3).$$

Then $DM = RD$ and $M(\ker D) \subseteq \ker D$. For

$$a = (1, 1, 0), \quad b = (1, 1, 17/10),$$

one has, for every integer $j \geq 0$,

$$D(M^j a) = D(M^j b) = (2^{-j}, 3^{-j}), \quad \|M^j b - M^j a\| = \frac{17}{10} \left(\frac{7}{10} \right)^j > 0.$$

Indeed, matrix multiplication gives $DM = RD$, induction gives $DM^j = R^j D$, and the difference occupies the third coordinate alone. The hidden differences at the first two events are exactly $17/10$ and $119/100$. The full wake retains that difference while its quotient evolves exactly.

For the momentum sector, closedness and pressure compatibility are supplied by the explicit cochain construction below. Successive inherited deposition enters the same event law through $r_k = D_{M,k} a_k^w$ and $a_{k+1}^w = M_k^w a_k^w$. This combines the quotient mechanism with the complete successive-wake history of Section 52.2.

Tables 2–4 reproduce the reported finite wake-quotient samples. The exact quotient criterion is Lemma 57.6; the internal-face operator is constructed in Section 57.8.

The reported tests evaluate the kernel condition of Lemma 57.6, the observational quotient morphism of Proposition 57.18, and the closed-cochain constraint. Their quantities are finite operators, cochains, and deposited wakes.

57.7.1. Non-injective projection and hidden wake ancestry

For the non-injective sample, the event- k wake space has dimension 3 while the deposition map has rank 2, leaving a one-dimensional hidden fiber. The paired initial states differ by norm 1.7 in the full wake space but have identical observed deposition. After one wake handoff the hidden states remain distinct, with successor difference norm 1.19, while their observed successor difference remains exactly zero. Thus distinct wake histories occupy the same observed quotient class at both events.

$$Y_1 \neq Y_2, \quad \Pi_{M,k}(Y_1) = \Pi_{M,k}(Y_2), \quad \Pi_{M,k+1}(E(Y_1)) = \Pi_{M,k+1}(E(Y_2)).$$

57.7.2. Event-dependent kernel transport and induced wake morphism

The operator identity required for the induced map is $D_{M,k+1} M_k^w = R_{M,k}^w D_{M,k}$. In the non-injective sample the kernel-preservation residual is exactly zero and the operator factorization residual is 1.189266×10^{-17} . The two representatives of the same event- k

Table 2. Non-injective wake quotient test. The vanishing observed differences with nonzero hidden-state differences are the defining numerical signature of the quotient fiber.

quantity	value
rank $D_{M,k}$	2
$\dim W_M$	3
$\dim \ker D_{M,k}$	1
hidden initial difference	1.7
observed initial difference	0.000×10^0
hidden successor difference	1.19
observed successor difference	0.000×10^0
kernel preservation residual	0.000×10^0
operator factorization residual	1.189×10^{-17}

quotient class therefore remain in a common successor class to numerical precision, with quotient-relation residuals 1.217400×10^{-17} for each representative.

$$M_k^w(\ker D_{M,k}) \subset \ker D_{M,k+1} \implies r_{k+1} = R_{M,k}^w r_k.$$

57.7.3. Observed wake-morphism factorization across successive events

The observed wake-morphism dataset records the factorization at two successive event transitions. The reported factorization residuals are 6.018531×10^{-36} and 4.369214×10^{-18} , while the corresponding successor-relation residuals are 1.734723×10^{-18} and 3.974751×10^{-18} . The deposited wake norms decrease from 0.0366 to 0.0305533 and 0.0267128 while the morphism remains closed. This is the event-dependent behavior required by the theorem: the observed wake space may change from event to event without requiring recovery of the hidden source state.

Table 3. Successive observed wake-morphism tests.

transition	factorization residual	successor residual	wake norm
$0 \rightarrow 1$	6.019×10^{-36}	1.735×10^{-18}	0.0305533
$1 \rightarrow 2$	4.369×10^{-18}	3.975×10^{-18}	0.0267128

57.7.4. Cochain closure and complete finite-event residual

The cochain sample verifies the discrete-complex identities that underlie the closed momentum sector and the finite-to-shadow reconstruction. The composition $d_1 d_0$ vanishes exactly in the sample. Initial and updated closedness residuals are 2.288475×10^{-17} and 2.955034×10^{-16} . The wake factorization and successor-relation residuals are 4.252574×10^{-18} and 4.249187×10^{-18} , respectively. The complete finite-event residual is 1.105355×10^{-15} . These values are at floating-point residual scale for the supplied calculation and are consistent with the exact cochain identities proved above.

Table 4. Discrete cochain and finite-event residual checks.

metric	norm
$d_1 d_0$	0.000×10^0
initial closedness	2.288×10^{-17}
updated closedness	2.955×10^{-16}
wake factorization	4.253×10^{-18}
wake successor relation	4.249×10^{-18}
finite event residual	1.105×10^{-15}
wake state change	0.292746

57.7.5. Interpretation

The reported samples illustrate a non-injective observation whose hidden kernel is transported into the successor hidden kernel. Distinct histories can therefore share both the present deposition and its successor. The cochain sample also records closedness of the event ledger. The exact quotient criterion is proved in Lemma 57.6; the spatial operators are constructed below.

This construction also connects the present fluid implementation to the finite-support admissibility and persistent-wake architecture developed in the companion mathematical-physics papers [15–17], while the Whitney/de Rham/Hodge components of the reconstruction are standard discrete geometric machinery [23–27].

57.8. Constructed finite spaces and nonzero flux

The following realization supplies the spaces and operators used in the cochain balance (57.45)–(57.46). The complete-state estimates are given in Corollary 55.4 and Theorem 55.6, according to the specified packet or inherited-readout recurrence.

Theorem 57.19 (Explicit compatible finite realization). Subdivide a periodic Cartesian grid on $\mathbb{T}^3$ into its six Freudenthal tetrahedra per cube, with at least three grid intervals in each direction. Give each simplex its incidence orientation. There are explicit finite spaces satisfying (57.11)–(57.26), positive Hodge matrices, a uniquely solvable mean-zero pressure equation, and an internal-face operator (57.35) with nonzero quadratic transport on X_M .

Proof. Use the usual incidence cochains. On a tetrahedron with barycentric coordinates λ_i , the basis form for the oriented r -face $[i_0 \cdots i_r]$ is

$$\omega_{i_0 \cdots i_r} = r! \sum_{j=0}^r (-1)^j \lambda_{i_j} d\lambda_{i_0} \wedge \cdots \wedge \widehat{d\lambda_{i_j}} \wedge \cdots \wedge d\lambda_{i_r}.$$

Assemble these with the global incidence signs. Integrating on oriented faces gives $I^r W^r = I$, and differentiating the displayed forms gives $dW^r = W^{r+1} d_r$. The traces agree on shared faces. These are the Whitney identities; see also [23,36]. Define H_r to be their exact L^2 Gram matrices. If $c^T H_r c = 0$, its reconstructed form vanishes, and $I^r W^r c = c$ gives $c = 0$. Hence every H_r is positive definite.

The two-form vector proxies are affine on each tetrahedron. Set Q_M equal to the coefficients of all discontinuous cellwise tensor polynomials of degree at most two, and let W^Q be their direct polynomial reconstruction. It is injective, and multiplication of the affine proxy coordinates defines B_M explicitly. Thus the quadratic range condition holds automatically.

Write $v_K = |K|$ for the tetrahedron volumes. The top Hodge matrix is $H_3 = \text{diag}(v_K^{-1})$. Every face meets exactly two cells with opposite incidence signs. Connectedness of the dual cell graph gives $\ker d_2^T = \text{span}\{\mathbf{1}\}$. Consequently

$$\ker \delta_3 = \text{span}\{(v_K)_K\} = \text{span}\{\mathbf{1}_M\}.$$

The pressure energy identity already proved makes $L_{p,M}$ invertible on the orthogonal complement of this vector. In particular

$$P_X = I - \delta_3 (L_{p,M}|_{C_{M,0}^3})^{-1} d_2$$

is the H_2 -orthogonal projection onto X_M . Every quantity is obtained by a finite matrix construction and polynomial integration.

For nonzero quadratic transport, partition the grid into three horizontal strips, indexed cyclically by $j = 0, 1, 2$, and put

$$u|_j = (f_j, 1, 0), \quad f = (0, 1, 0), \quad v|_j = (h_j, 0, 0), \quad h = (1, 0, -1).$$

These fields have continuous normal components across every face and zero cellwise divergence, so they are Whitney reconstructions of closed cochains m, v_M . Their

volume-gradient pairings vanish. Internal tetrahedral faces inside a strip also give zero jump; summing the strip interfaces, each of total area one, gives

$$\langle C_M B_M(m, m), v_M \rangle_{H_2} = \sum_{j=0}^2 \frac{f_j + f_{j+1}}{2} (h_j - h_{j+1}) = 1.$$

As $v_M \in X_M$, this proves $P_X C_M B_M(m, m) \neq 0$. The pressure cannot remove this transport. $\square$

Lemma 57.20 (Closed-sector skew transport). For the preceding lowest-order Whitney realization and $b \in X_M$, define

$$L_b n = P_X C_M B_M(n, b), \quad n \in X_M.$$

Then L_b is skew-adjoint in the H_2 inner product.

Proof. On an affine tetrahedron a lowest-order Whitney two-form has vector proxy $a + cx$ with scalar c . Closedness gives $3c = 0$, so every closed proxy is cellwise constant. Its normal component agrees across faces. For closed n, v, b , the face pairing is therefore

$$\langle L_b n, v \rangle_{H_2} = \sum_F \int_F (u_b \cdot n_F) \{u_n\}_F \cdot [u_v]_F dS.$$

Adding the expression with n, v exchanged uses $\{u_n\} \cdot [u_v] + \{u_v\} \cdot [u_n] = [u_n \cdot u_v]$. The sum becomes

$$\sum_K (u_n|_K \cdot u_v|_K) \int_{\partial K} u_b \cdot n_K dS = 0,$$

by closedness of b . This proves skew-adjointness and allows nonzero $L_m m$, as the preceding witness demonstrates. $\square$

Remark 57.21 (Momentum submap and complete-state work). For the internal-face operator above, set wake deposition, forcing, and viscosity to zero and use the fixed H_2 momentum norm. Then $m^+ = m - \delta L_m m$, and skew-adjointness gives

$$\|m^+\|_{H_2}^2 = \|m\|_{H_2}^2 + \delta^2 \|L_m m\|_{H_2}^2.$$

The spatial witness above makes the final term nonzero. This is the work identity of the indicated momentum submap. The complete retained event has the live-channel work identity (53.5), including residual returns and transported geometry.

For a compatible linear momentum readout $m_k = \mathcal{C}_k x_k$ of a complete retained state, the norm transfer is explicitly

$$\|m_k\|_{H_2} \leq \|\mathcal{C}_k\|_{k \rightarrow H_2} \|x_k\|_k.$$

Consequently a uniform complete-state estimate gives a uniform momentum estimate whenever the actual readout norms are uniformly bounded. Both the event compatibility and these norms are properties of the specified map $\mathcal{C}_k$.

Proposition 57.22 (Finite ternary implementation of the wake construction). Choose rational carrier coordinates, $\kappa = 3/5$, $d = 4/5$, the exchange matrix of (51.7), and rational isometric wiring in a fixed carrier. With rational initial data and rational, explicitly supplied remainder packets, each finite prefix of the recurrence in Corollary 55.4 is produced by exact arithmetic instructions using at most two input registers and one output register.

Proof. The completed aperture has rational entries. The Wallis–Gregory coefficients are finite rational products and sums. Each matrix product and packet update therefore reduces to additions, subtractions, multiplications, and divisions by the displayed nonzero denominators. Ordering those scalar instructions gives the stated finite implementation. The exact memory formula is obtained by the same finite substitutions and needs no independent forcing registers for inherited returns. $\square$

This proves a three-register arithmetic implementation of the specified wake construction. An assertion about physical three-body interactions retains the separate content of Assumption 63.2.

58. Boundary-inclusive finite event law and wall-bounded extension

The differential-form and orientation conventions are collected in [28].

The periodic theorem isolates the algebra of the compatible reconstruction. Physical walls can be included without changing the finite-first ontology by retaining boundary traces as finite cochain data rather than discarding them by periodicity.

58.1. Orientable polyhedral wall domains

Let Ω be a compact orientable polyhedral three-manifold with boundary and let K_M^∂ be an oriented finite complex resolving $\partial\Omega$. Let γ_M denote the compatible boundary trace on reconstructed fields and let B_M be a finite boundary-traction cochain space with injective trace reconstruction $W_{\partial,M}^B$.

Define the boundary-inclusive tensor-divergence pullback $C_M^\partial : Q_M \rightarrow C_M^2$ by

$$\begin{aligned} \langle C_M^\partial q, v \rangle_{H_2} = & - \sum_K \int_K Q : \nabla_h u_v \, dx + \sum_{F \in \mathcal{F}_M^{\text{int}}} \int_F (\{Q\}_F n_F) \cdot [u_v]_F \, dS \\ & + \int_{\partial\Omega} (Qn) \cdot u_v \, dS. \end{aligned} \quad (58.1)$$

where $Q = W_M^Q q$ and $u_v = P_M^u v$. The internal-face term uses the same orientation convention as (57.35). The final term is the resolved exterior tensor-flux trace already contained in Q ; it is not counted again as an external traction.

For an additional physical wall traction $b_k \in B_M$, define its volume-ledger pullback $T_{\partial,M} : B_M \rightarrow C_M^2$ by

$$\langle T_{\partial,M} b, v \rangle_{H_2} := \int_{\partial\Omega} W_{\partial,M}^B b \cdot \gamma_M u_v \, dS. \quad (58.2)$$

Let G_M^∂ and A_M^∂ denote, respectively, the pressure multiplier and discrete Hodge Laplacian on the chosen boundary-constrained cochain spaces. Their domains encode the stated wall condition, and the pressure gauge is again mean zero. The one-event balance is

$$\frac{m_{k+1} - m_k}{\Delta\tau_k} + C_M^\partial q_k + G_M^\partial \pi_k + \nu_M A_M^\partial m_k = g_k + T_{\partial,M} b_k. \quad (58.3)$$

The pressure solvability hypothesis is stated explicitly as

$$d_2(C_M^\partial q_k - g_k - T_{\partial,M} b_k) \in \text{Ran}(d_2\delta_3^\partial), \quad (58.4)$$

with the constant harmonic mode removed whenever it is present.

Theorem 58.1 (Conditional trace-compatible wall commutation). This theorem is an extension schema for any chosen boundary discrete de Rham/Hodge complex whose interior and trace reconstructions satisfy the stated commuting properties. Assume compatible Whitney reconstruction in the interior, compatible trace-Whitney reconstruction on $\partial\Omega$, specified operators G_M^∂ and A_M^∂ on the boundary-constrained cochain spaces, and (58.4). Then reconstruction of (58.3) gives the exact continuum projected affine wall balance

$$\partial_t u_M + \text{Div}_M^\partial Q_M^k + \text{Grad}_M^\partial p_M^k - \nu_M \Delta_M^\partial u_M^k = f_M^k + T_{\partial,M}^k b_k \quad (58.5)$$

for almost every affine observation point in the completed event interval, and the corresponding boundary commutation defect vanishes.

Proof. Apply the interior Whitney commuting identities and the trace commuting identities to each term of (58.3). The boundary Riesz pullbacks (58.1) and (58.2) reproduce the surface pairings exactly. The pressure compatibility condition determines the gauge-fixed multiplier preserving the boundary-constrained closed-flux sector, while the boundary Hodge operator reconstructs by definition with the stated diffusion sign. Term-by-term equality gives exact commutation.

Remark 58.2 (Boundary operators and traces). Theorem 58.1 uses the boundary discrete de Rham complex, boundary Hodge star, pressure space, and commuting trace maps. Theorem 58.3 constructs these objects for impermeable walls on a relative two-cochain sector with zero boundary-face flux. A no-slip trace also prescribes the tangential boundary degrees of freedom introduced below.

For an impermeable slip wall, the admissible momentum space is restricted by vanishing normal flux,

$$X_M^{\text{slip}} := \{m \in \ker d_2 : \text{tr}_{n,M} m = 0\}, \quad (58.6)$$

and tangential velocity remains free subject to the chosen tangential traction law. A genuine no-slip wall requires additional tangential boundary degrees of freedom. Let

$$h_k \in C^1(\partial K_M; \mathbb{R}) \quad (58.7)$$

be a tangential boundary impulse/circulation cochain. Its trace reconstruction is coupled to the finite circulation ledger, so the wall contribution to circulation is represented at finite resolution by

$$\Gamma_{k+1}^\partial - \Gamma_k^\partial = \langle h_k, \partial P_k \rangle \quad (58.8)$$

before any smooth boundary-layer description is invoked. No-slip is imposed by the compatible trace constraint $\gamma_M u_M = 0$; wall vorticity generation is then recorded by the tangential cochain rather than inferred from an infinitesimal layer.

Theorem 58.3 (Explicit impermeable-wall realization). Let a connected compact orientable polyhedral three-manifold with boundary have a finite conforming tetrahedral triangulation and connected dual cell graph. Use the relative cochain complex $C^r(K, \partial K)$, the restricted Whitney basis, its exact L^2 Hodge matrices, and the same cellwise quadratic tensor space. These choices discharge the interior, trace, positivity, and pressure-compatibility hypotheses of Theorem 58.1 for impermeable walls with free tangential trace and specified finite traction data.

Proof. Relative cochains vanish on boundary simplices. The simplicial trace commutes with d , so they form a subcomplex; restricting the explicit Whitney forms gives the corresponding commuting trace identities. For two-forms, zero pullback to boundary faces is exactly zero normal velocity. The restricted Gram matrices remain positive definite because $I^r W^r = I$ on the relative degrees of freedom.

The relative d_2 has the interior faces as its columns. Each column joins two cells with opposite signs, so connectedness gives $\ker d_2^T = \text{span}\{\mathbf{1}\}$. With $H_3 = \text{diag}(|K|^{-1})$, its harmonic top vector is again $(|K|)_K$. The pressure energy argument proves invertibility on the mean-zero subspace. Every right-hand side $d_2 y$, including $y = C_M^\partial q - g - T_{\partial, M} b$, is orthogonal to that harmonic vector and hence lies in $\text{Ran}(d_2 \delta_3)$. This proves (58.4) instead of assuming it.

All boundary pairings are integrals of explicit polynomial traces and therefore define finite Riesz maps. For a closed advector with zero normal wall trace, the exterior advective flux is zero; the proof of Lemma 57.20 has no remaining boundary contribution. The wake-channel identities remain applicable when the wall event retains and wires those channels with their stated metrics; the relative complex supplies the spatial pressure and trace operators. A no-slip tangential constraint or a specified continuum traction law requires its own additional realization; neither is supplied by the zero-normal-flux condition alone. $\square$

58.2. Non-orientable Möbius interfaces and twisted cochains

A Möbius strip cannot be a connected component of the boundary of an orientable three-dimensional fluid region. Accordingly, a Möbius “wall” is treated here as an internal non-orientable material interface Σ , or as an abstract non-orientable wall manifold, rather than as an ordinary component of $\partial\Omega$.

Let o_Σ be the orientation line bundle of Σ . Boundary/interface flux and traction data are then represented by twisted cochains

$$C^r(\Sigma; o_\Sigma), \quad (58.9)$$

so that the globally meaningful object is the twisted density $(Qn)dS$, even though no global choice of n exists. The weak interface pairing is therefore well defined without choosing a global normal.

Proposition 58.4 (Constructed two-sector lift and compatible holonomy metric). The orientation local system of a Möbius interface has character -1 on its reversing generator γ . Define a rank-two local system by the representation of $\pi_1(\Sigma) \cong \mathbb{Z}$ that sends γ to

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (58.10)$$

Its locally labelled coordinate lines are exchanged, the Euclidean metric is preserved, and its determinant local system satisfies

$$\det E_\Sigma \simeq o_\Sigma. \quad (58.11)$$

Thus such a balanced lift exists by construction.

More generally, let a prescribed monodromy H exchange two transverse lines and have determinant -1 as a local-system character. In a branch basis $H = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ with $ab = 1$, so $H^2 = I$. Starting with any positive metric $g_\perp$ making the lines orthogonal, set

$$g_H(u, v) = \frac{1}{2} \{g_\perp(u, v) + g_\perp(Hu, Hv)\}.$$

Then H preserves g_H . Choosing $e \in L^+$ unit and $f = He \in L^-$ gives the orthonormal branch basis in which $H = C$.

Proof. The first construction is a representation because every power of the generator is sent to the corresponding power of C ; its determinant is $(-1)^n$ on γ^n . For the second assertion, $ab = 1$ proves involutivity. Applying H interchanges the two summands of g_H , and the exchanged-line property preserves orthogonality. Thus $He = f$ and $Hf = e$ in a unit branch basis. A fixed unrelated metric need not have this property. The explicit lift and the averaged metric discharge existence and normalization for this constructed class. $\square$

Topology alone does not imply nonzero circulation or perpetual vorticity production: the zero twisted cochain is always admissible. A nonzero topological contribution requires an occupied holonomy channel together with nonzero interface traction, wake, or circulation data. The Möbius geometry therefore constrains how sign and circulation are transported; it does not create momentum or energy from nothing.

59. Finite wake absorption and observational invisibility

A retained wake need not vanish in order to disappear as a separate term in the projected momentum equation.

Proposition 59.1 (Pressure-absorbable finite wake). Suppose that for event k there exists $\chi_k \in C_{M,0}^3$ such that

$$C_M r_k = G_M \chi_k. \quad (59.1)$$

Then, with

$$\pi_k^\sharp := \pi_k + \chi_k, \quad (59.2)$$

the exact finite balance can be written

$$\frac{m_{k+1} - m_k}{\Delta \tau_k} + C_M B_M(m_k, m_k) + G_M \pi_k^\sharp + \nu_M A_M m_k = g_k. \quad (59.3)$$

Thus a nonzero inherited wake can remain present in the finite ontology while being absorbed into the projected pressure channel. In the stronger special case $C_M r_k = 0$, the wake is completely invisible to the projected momentum balance at resolution M .

Proof. Insert $q_k = B_M(m_k, m_k) + r_k$ into the one-event law and use (59.1). The two gradient terms combine linearly:

$$C_M r_k + G_M \pi_k = G_M \chi_k + G_M \pi_k = G_M (\pi_k + \chi_k).$$

This is the same sign convention used in the smooth-shadow statement below: the finite gradient is written on the left, while the corresponding pressure gradient appears with a minus sign after moving it to the right.

60. Circular vector projection and the classical fluid equation

Classical fluid notation in this realization follows [29–31].

The native history and its source-clock continuation are determined by Theorem 57.11. Circular carriers give an explicit smooth vector readout. The smoothness follows from the finite projection formula itself, with the same event-clock coordinate used throughout.

Lemma 60.1 (Smoothness of finite circular vector projections). Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

For finitely many indices j , fix real linear maps $B_j : \mathbb{R}^2 \rightarrow \mathbb{R}^d$, vectors $v_j \in \mathbb{R}^2$, wave vectors $k_j \in \mathbb{R}^n$, frequencies $\omega_j \in \mathbb{R}$, and $\gamma_j \geq 0$. Then the projected field

$$F(x, t) = b + \sum_j B_j e^{-\gamma_j t} R(k_j \cdot x - \omega_j t) v_j \quad (60.1)$$

is C^∞ for every $t \geq 0$. For $a + |\beta| \geq 1$,

$$\|\partial_t^a \partial_x^\beta F(x, t)\| \leq \sum_j \|B_j\| \|v_j\| (\gamma_j + |\omega_j|)^a \|k_j\|^{|\beta|} e^{-\gamma_j t}.$$

When $d = n$ and $k_j^T B_j = 0$ for each j , the field is divergence free.

Proof. The identity $R'(\theta) = JR(\theta)$ gives every derivative of each summand. A time derivative contributes $-\gamma_j I - \omega_j J$, whose norm is at most $\gamma_j + |\omega_j|$; a spatial derivative contributes the appropriate component of k_j and the norm-one matrix J . There are finitely many summands, so termwise differentiation is exact at every order and gives the stated bound. The divergence of the j th summand contains the factor $k_j^T B_j$, and therefore is zero under the stated transversality condition. $\square$

The line component $e_1^T R(\theta) v$ is a linear projection of a circular carrier, and the maps B_j assemble such components into a vector field. The derivative bound proves regularity directly for this finite vector projection. Proposition 60.4 supplies an exact finite successor law whose circular projection also satisfies the nonlinear classical fluid equation.

Definition 60.2 (Compatible classical readout). A compatible classical readout of an occupied history with wake-realization coordinate $\tau = \Theta_w(X_{\leq \cdot})$ is a reconstruction

$$(u, p, f) = P_{\text{cl}}(X)$$

with $u \in C_t^1 C_x^2$, $p \in C_t^0 C_x^1$, $f \in C^0$, where the symbol t denotes the same wake-realization coordinate used in (57.72), now represented smoothly rather than eventwise affinely, such that

$$\nabla \cdot u = 0, \quad \text{Grad}_{\text{cl}} p = \nabla p, \quad \Delta_{\text{cl}} u = \Delta u, \quad (60.2)$$

and the reconstructed complete second moment satisfies

$$Q_{\text{cl}} = u \otimes u + \tau_{\text{cl}}, \quad \text{Div}_{\text{cl}} Q_{\text{cl}} = \nabla \cdot Q_{\text{cl}}. \quad (60.3)$$

The smooth readout is required to commute with the finite event balance on the occupied history.

Applying the compatible readout to the retained-stress balance gives

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f - \nabla \cdot \tau_{cl}, \quad \nabla \cdot u = 0. \quad (60.4)$$

A retained stress contributes to the pressure channel whenever

$$\nabla \cdot \tau_{cl} = \nabla \chi, \quad p^\sharp = p + \chi. \quad (60.5)$$

Writing the absorbed pressure as p in the classical equation gives

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \quad \nabla \cdot u = 0. \quad (60.6)$$

The stress and its inherited state remain present. The following proposition characterizes this pressure absorption with the forcing fixed. In subsequent classical identities, p denotes the absorbed pressure readout.

Proposition 60.3 (Exact pressure absorption of smooth retained stress). Consider a smooth periodic realization of (60.4) on a flat torus, with the forcing f fixed. Let $\mathbb{P}$ denote its orthogonal solenoidal projection. The same velocity and forcing satisfy ordinary incompressible Navier–Stokes with a pressure $p^\sharp$ if and only if

$$\mathbb{P} \nabla \cdot \tau_{cl} = 0. \quad (60.7)$$

In that case,

$$\nabla \cdot \tau_{cl} = \nabla \chi, \quad p^\sharp = p + \chi,$$

where χ is the mean-zero periodic solution of $\Delta \chi = \nabla \cdot (\nabla \cdot \tau_{cl})$. The retained stress itself may be nonzero.

Proof. The periodic vector field $g = \nabla \cdot \tau_{cl}$ has mean zero. Its Fourier decomposition gives $g = \mathbb{P}g + \nabla \Delta^{-1} \nabla \cdot g$. Thus (60.7) is equivalent to $g = \nabla \chi$ with the displayed χ . Substitution in (60.4) combines the two gradient terms as $-\nabla(p + \chi)$ and gives the ordinary equation with the same u, f . Conversely, subtraction of that ordinary equation from (60.4) gives $g = \nabla(p^\sharp - p)$, whose solenoidal projection vanishes. $\square$

For example, $\tau_{cl} = h(t, x)I$ gives $\nabla \cdot \tau_{cl} = \nabla h$. More generally, $\tau_{cl} = hI + S$ with $\nabla \cdot S = 0$ has the same pressure absorption. These retained states need not vanish or lose their ancestry when their stress contribution is absorbed into the pressure readout.

Define $\omega = \nabla \times u$ and $B = p + |u|^2/2$. The Lamb identity

$$(u \cdot \nabla)u = \nabla \left(\frac{|u|^2}{2} \right) - u \times \omega \quad (60.8)$$

gives

$$\partial_t u + \omega \times u = -\nabla B + \nu \Delta u + f. \quad (60.9)$$

The gradient part and rotational transport are therefore separated exactly in the classical shadow.

For a material loop $C(t)$ parametrized by the same smooth wake-realization coordinate,

$$\Gamma_C(t) = \oint_{C(t)} u \cdot d\ell \quad (60.10)$$

satisfies

$$\frac{d\Gamma_C}{dt} = \nu \oint_{C(t)} \Delta u \cdot d\ell + \oint_{C(t)} f \cdot d\ell. \quad (60.11)$$

For Euler flow ($\nu = 0$, $f = 0$), Γ_C is constant. Stokes' theorem gives

$$\Gamma_C(t) = \int_{S(t)} \omega \cdot n \, dA. \quad (60.12)$$

Hence vortex amplification in the smooth shadow may occur through deformation of an inherited circulation carrier rather than independent creation of circulation.

Proposition 60.4 (Global classical fluid from circular vector projection). Fix $c \in \mathbb{R}$, $\nu \geq 0$, an integer $J_0 \geq 1$, a real constant a_0 , and vectors $v_j^0 \in \mathbb{R}^2$ for $1 \leq j \leq J_0$. Put $\kappa_j = 2\pi j$ and $t_0 = 0$. At the occupied source-clock endpoints choose $\delta_k = t_{k+1} - t_k \geq \delta_* > 0$ and use the finite event

$$v_j^{k+1} = e^{-\nu \kappa_j^2 \delta_k} R(-c \kappa_j \delta_k) v_j^k. \quad (60.13)$$

On $[t_k, t_{k+1}]$, use the same formula with δ_k replaced by $t - t_k$. Define on the unit three-torus

$$U(y, t) = a_0 + 2 \sum_{j=1}^{J_0} e_1^T R(\kappa_j y) v_j(t), \quad u(x, y, z, t) = (U(y, t), c, 0). \quad (60.14)$$

Let h be any scalar finite circular projection of the form in Lemma 60.1, with periodic spatial modes. Retain its coefficient vectors z_ℓ^k in the event state and update them by $z_\ell^{k+1} = e^{-\gamma_\ell \delta_k} R(-\omega_\ell \delta_k) z_\ell^k$. Set

$$\tau_{\text{cl}} = hI, \quad p = -h, \quad f = 0.$$

Then u is globally C^∞ and incompressible, and it satisfies the wake-resolved equation (60.4). With $p^\sharp = p + h = 0$, it satisfies ordinary Navier–Stokes for all $t \geq 0$. The stress can be nonzero throughout. The finite circular event and its smooth vector readout commute exactly. When $c \neq 0$ and a nonconstant mode is occupied, $(u \cdot \nabla)u$ is nonzero.

Proof. The identities $R(\alpha)R(\beta) = R(\alpha + \beta)$ and $e^{-\gamma\alpha}e^{-\gamma\beta} = e^{-\gamma(\alpha+\beta)}$ give

$$v_j(t) = e^{-\nu \kappa_j^2 t} R(-c \kappa_j t) v_j^0$$

on every event interval. Consequently adjacent vector projections agree with all derivatives at the common endpoint. The positive step floor makes the intervals cover all nonnegative times. Lemma 60.1 proves global smoothness directly.

Since $R'(\theta) = JR(\theta)$ and $J^2 = -I$, differentiation gives

$$U_t + cU_y = \nu U_{yy}, \quad \nabla \cdot u = 0, \quad (u \cdot \nabla)u = (cU_y, 0, 0), \quad \Delta u = (U_{yy}, 0, 0).$$

Also $-\nabla p - \nabla \cdot (hI) = \nabla h - \nabla h = 0$. Substitution proves the wake-resolved equation and its pressure-absorbed classical form. At each endpoint the reconstruction is exactly the projection of (60.13). Linear independence of the Fourier modes shows $U_y \not\equiv 0$ when a nonconstant mode is occupied. $\square$

This construction proves classical smoothness and exact event commutation for the stated finite circular family. The inherited wake, the pressure readout, and the vector projection are specified parts of the event state. For the cochain balance (57.45), Definition 60.2 specifies how the classical observation acts on the momentum and complete wake.

61. Classical vorticity propagator and the DZE overlap

This section is entirely within the strong smooth-shadow subclass. The variable t below is the smooth realization of the same wake-realization coordinate fixed by Θ_w ; it is not a second external clock. Taking the curl of (60.6) yields

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)u + \nu \Delta \omega + \nabla \times f. \quad (61.1)$$

Let $S = (\nabla u + \nabla u^T)/2$. Since the antisymmetric part of ∇u annihilates ω when acting on ω itself,

$$(\omega \cdot \nabla)u = S\omega. \quad (61.2)$$

Along a material trajectory $X(a, t)$ define the fundamental matrix $G(t, s)$ by

$$\partial_t G(t, s) = S(X(a, t), t)G(t, s), \quad G(s, s) = I. \quad (61.3)$$

Theorem 61.1 (Classical vorticity ancestry). Along every smooth material trajectory,

$$\omega(X(a, t), t) = G(t, 0)\omega_0(a) + \int_0^t G(t, s)[\nu \Delta \omega + \nabla \times f](X(a, s), s) ds. \quad (61.4)$$

Proof. Equation (61.1) restricted to a material path is an inhomogeneous linear equation in the instantaneous vorticity vector with coefficient matrix $S(X(a, t), t)$. Variation of constants gives (61.4).

The classical shadow therefore already has the same architecture as (56.12): a propagated ancestral rotational state plus all later inputs propagated from their entry times.

The finite-support relocking law used in the DZE construction has the form

$$\frac{dA}{ds} = \Lambda_1 D(s) - \Lambda_2 A(s), \quad \Lambda_2 > 0, \quad (61.5)$$

and therefore

$$A(s) = e^{-\Lambda_2(s-s_0)} A(s_0) + \Lambda_1 \int_{s_0}^s e^{-\Lambda_2(s-\sigma)} D(\sigma) d\sigma. \quad (61.6)$$

Proposition 61.2 (Propagator equivalence). Equations (56.12), (61.4), and (61.6) share the same algebraic skeleton:

$$\text{state} = \text{propagated ancestral state} + \text{history input propagated from its point of entry}. \quad (61.7)$$

This is a structural equivalence of propagator form, not an assertion that the physical variables in the three equations are identical.

62. Why the three-dimensional Jacobian does not close the dynamics

The determinant and pullback identities use the tensor conventions in [32].

For the classical flow map $X(a, t)$ let

$$F(a, t) = \nabla_a X(a, t), \quad \dot{F} = (\nabla u)F. \quad (62.1)$$

Jacobi's formula gives

$$\frac{d}{dt} \det F = (\nabla \cdot u) \det F. \quad (62.2)$$

Thus incompressibility and $F(a, 0) = I$ imply

$$\det F = 1. \quad (62.3)$$

But

$$F_\Lambda = \text{diag}(\Lambda, \Lambda^{-1}, 1) \quad (62.4)$$

has $\det F_\Lambda = 1$ while

$$\text{cond}(F_\Lambda) = \Lambda^2 \longrightarrow \infty. \quad (62.5)$$

Therefore

$$\det F = 1 \not\Rightarrow \text{bounded directional distortion}. \quad (62.6)$$

A scalar determinant preserves volume but cannot reconstruct the sign carrier, anisotropic loading, or wake ancestry erased by a non-injective projection.

63. Nested three-body closure instead of many-body Gaussian collapse

The finite-support construction does not replace a complicated interaction by independent Gaussian kicks. Instead, the interaction is resolved by an ordered triadic cover.

Definition 63.1 (Three-body closure event). A local event is a triple

$$\mathcal{T}_j = (S_j, R_j, M_j), \quad (63.1)$$

where S_j is the advancing/source body, R_j is the responding body, and M_j is the mediating interface or environment. The event produces a retained wake K_j and an outgoing carrier V_{j+1} according to

$$V_{j+1} = T_j V_j + K_j, \quad K_j = W_j(\mathcal{T}_j; K_{j-1}, \dots, K_0). \quad (63.2)$$

The second relation allows the wake law to depend on the inherited history.

Assumption 63.2 (Ordered triadic cover). The occupied many-body history admits a finite ordering of local closures $\mathcal{T}_0, \dots, \mathcal{T}_{N-1}$ such that every interaction actually used by the occupied trajectory enters through one of these triads and the outgoing state of a completed event supplies data to the next event.

Theorem 63.3 (Exact reduction of an occupied many-body history). Under Assumption 63.2, the stage- N carrier is exactly

$$V_N = T_{N-1} \cdots T_0 V_0 + \sum_{j=0}^{N-1} T_{N-1} \cdots T_{j+1} W_j(\mathcal{T}_j; K_{j-1}, \dots, K_0). \quad (63.3)$$

No independent-noise replacement is required.

Proof. Equation (63.3) is (56.12) with the event wake written as the history-conditioned output of its local three-body closure. Because the ordered cover contains every interaction used by the occupied history, the finite substitution is exact on that history.

If additionally

$$B_{j+1} T_j = \lambda_j O_j B_j, \quad (63.4)$$

then the contribution injected at stage j carries the inherent later scale factor

$$\left(\prod_{k=j+1}^{N-1} \lambda_k \right) O_{N-1} \cdots O_{j+1} \quad (63.5)$$

inside the corresponding coordinate frame. Scaling is therefore generated by the same transport that carries the wake; it is not added as an external statistical correction.

64. Separate p - q wake scaling on intrinsic Hausdorff support

The statistical profiles are expressed in the notation of [33].

The intrinsic geometry from Section 50 and the transport/statistical readouts must be kept distinct. We use the triple

$$\Theta_{\text{wake}} := (D_H, p, q) \quad (64.1)$$

as a descriptor of a realized wake regime:

- D_H describes the occupied sign-carrier geometry;
- p describes transport/intermittency weighting through changing support;
- q describes the nonextensive distributional law of retained wake excursions.

No universal algebraic identity $q = q(p)$ or $q = q(D_H)$ is assumed.

For $\beta > 0$ and $1 < q < 3$, define the normalized density on $\mathbb{R}$ by

$$P_q(z) = Z_q^{-1} [1 + (q-1)\beta z^2]^{-1/(q-1)}, \quad Z_q = \sqrt{\frac{\pi}{(q-1)\beta}} \frac{\Gamma(\frac{1}{q-1} - \frac{1}{2})}{\Gamma(\frac{1}{q-1})}. \quad (64.2)$$

Proposition 64.1 (Normalization, Gaussian limit, and finite-support extension). The integral defining Z_q is finite exactly for $1 < q < 3$. In that range,

$$\lim_{q \downarrow 1} P_q(z) = \sqrt{\frac{\beta}{\pi}} e^{-\beta z^2}, \quad (64.3)$$

and, with $a = 1/(q-1)$,

$$P_q(z) \sim Z_q^{-1} [(q-1)\beta]^{-a} |z|^{-2a}. \quad (64.4)$$

For a fixed finite cutoff $L > 0$, the truncated density

$$P_{q,L}(z) = \frac{\mathbf{1}_{[-L,L]}(z) [1 + (q-1)\beta z^2]^{-1/(q-1)}}{\int_{-L}^L [1 + (q-1)\beta t^2]^{-1/(q-1)} dt}$$

is well defined for every $q > 1$.

Proof. Substitute $t = \sqrt{(q-1)\beta} z$ and then $t = \tan \theta$ in the full-line integral. The beta integral gives the displayed gamma quotient when $a > 1/2$. The tail is a positive constant times $|z|^{-2a}$, so this condition is also necessary. For the limit, the unnormalized density converges pointwise to $e^{-\beta z^2}$. For $0 < q-1 \leq 1/2$ it is bounded by $(1 + \beta z^2/2)^{-2}$: $\log(1 + \varepsilon t)/\varepsilon$ decreases in $\varepsilon > 0$. Dominated convergence gives $Z_q \rightarrow \sqrt{\pi/\beta}$ and proves (64.3). The asymptotic with its coefficient follows by factoring out $(q-1)\beta z^2$. On $[-L, L]$ the integrand is positive and continuous for every $q > 1$, so its integral is finite and nonzero. $\square$

For $p > 0$, a convenient unnormalized p -weighted transport kernel is

$$Q_p(\theta' | \theta) = Q_G(\theta' | \theta) (1 + p|\Delta H_{\parallel}|)^{-1/p}, \quad (64.5)$$

where $\Delta H_{\parallel}$ measures the longitudinal change in the projected curvature readout. The factor equals one when $\Delta H_{\parallel} = 0$ irrespective of $p > 0$. When a transition probability is required, divide the right-hand side by its integral in θ' , provided the base kernel is a probability density; this integral lies in $(0, 1]$ for finite $|\Delta H_{\parallel}|$ almost everywhere.

Definition 64.2 (Affine-shadow calibration). The straight/Gaussian shadow is the joint limit

$$D_H \rightarrow 1, \quad q \rightarrow 1, \quad \Delta H_{\parallel} \rightarrow 0. \quad (64.6)$$

The parameter p need not be identified with either D_H or q ; in the straight limit the p -weight becomes inactive because $\Delta H_{\parallel} \rightarrow 0$.

Thus the non-Gaussian wake is not obtained by sending a many-body problem into a Gaussian bath. The Gaussian law is instead the quadratic/lineal shadow obtained when the persistent support and its transport deformation are suppressed by projection.

65. Kolmogorov compatibility in the continuum shadow

The classical scaling comparison is with [37].

The $-5/3$ law is included as a compatibility statement of the projected inertial-range field, not as the definition of the primary wake geometry.

Theorem 65.1 (Kolmogorov shadow scaling). Assume that, over an inertial range of the continuum readout, velocity increments obey

$$|\delta u_{\ell}| \sim \ell^h \quad (65.1)$$

and that the mean energy flux through scale ℓ is scale independent:

$$\varepsilon \sim \frac{|\delta u_{\ell}|^3}{\ell} \sim \ell^{3h-1} \sim \text{constant}. \quad (65.2)$$

Then

$$h = \frac{1}{3}. \quad (65.3)$$

If the second-order structure function satisfies

$$S_2(\ell) = \langle |\delta u_{\ell}|^2 \rangle \sim \ell^{\zeta_2}, \quad \zeta_2 = 2h, \quad (65.4)$$

and $E(k) \sim k^{-m}$ with the standard inertial-range relation $\zeta_2 = m - 1$, then

$$E(k) \sim k^{-5/3}. \quad (65.5)$$

Proof. From $|\delta u_{\ell}| \sim \ell^h$, $\varepsilon \sim |\delta u_{\ell}|^3/\ell \sim \ell^{3h-1}$. Scale independence therefore requires $3h - 1 = 0$, so $h = 1/3$. Hence $\zeta_2 = 2h = 2/3$, and the relation $\zeta_2 = m - 1$ gives $m = 1 + 2/3 = 5/3$.

No identification of $\zeta_2 = 2/3$ with the Hausdorff excess χ_H , the p -transport exponent, or the q -statistical index is required. Any such identification would need an additional theorem.

66. Global native continuation and retained-state estimates

The native finite conclusion is already unconditional within the defined event class: Theorem 57.9 gives a unique history at every finite depth, and Corollary 57.17 gives exact observation with the entire inherited wake retained. The endpoint and projected-affine momentum identities follow without a wake quotient condition. Theorems 57.1 and 57.11 extend continuation through every finite interval of the anticonformal source clock at fixed resolution, without a uniform-in-index contraction estimate. The following synthesis adds the separate uniform estimates and specified carrier readouts to those finite results.

Theorem 66.1 (Native dynamics and compatible observations). Assume:

- (i) a finitely branching admissible refinement system (50.1) with intrinsic carrier L_∞ defined by (50.8);
- (ii) a balanced transverse carrier realization Φ_{car} as in (51.2);
- (iii) exact wake handoff and canonical–residual recombination;
- (iv) the Wallis–Gregory complete-state recurrence with nonexpansive handoffs and packets of norm at most E_w ; for the inherited-readout recurrence of Theorem 55.6, Corollary 55.7 supplies this packet budget from the initial state;
- (v) transported finite-vortex geometry (56.1)–(56.8), with canonical edge readouts of the complete retained carrier states as specified before (56.11);
- (vi) an ordered triadic cover of the occupied many-body history;
- (vii) a strictly increasing wake-realization clock map (57.1); and
- (viii) the compatible Whitney event model of Section 57, including the finite quadratic map B_M , augmented state (57.39), wake-closed deposition (57.29), faithful momentum readout of Proposition 57.4, derived pressure equation (57.44), and one-event balance (57.45); and
- (ix) the fully instantiated periodic geometry of Section 57.1.

Then, for every finite depth N and every admissible event k :

- (a) the complete retained state obeys

$$\|x_N\|_N \leq \frac{3\pi}{8} \|x_0\|_0 + \frac{E_w \pi}{96}.$$

For the inherited orthogonal-projection recurrence the sharper estimate is

$$\|x_N\|_N \leq \frac{73\pi}{192} \|x_0\|_0;$$

- (b) the finite vortex density obeys

$$|\beta_N^{\text{fin}}| \leq \frac{L_0}{\sqrt{A_0}} \left(\frac{3\pi}{8} S_0 + \frac{E_w \pi}{96} \right).$$

When every retained edge state follows the inherited orthogonal-projection recurrence, this becomes

$$|\beta_N^{\text{fin}}| \leq \frac{L_0}{\sqrt{A_0}} \frac{73\pi}{192} S_0;$$

- (c) the finite vortex has the exact ancestry decomposition (56.12);
- (d) finite circulation equals the circulation and integrated vorticity of the Whitney observational reconstruction by (56.16)–(56.17);
- (e) every apparent many-body contribution on the occupied trajectory is represented by the nested three-body history, with inherited scaling carried by subsequent transport maps;
- (f) the finite momentum event obeys the exact one-step law (57.45), the projected endpoint states obey (57.70), and the continuum projected affine observation obeys (57.75);
- (g) the observational velocity is exactly divergence free, pressure is derived by the finite constraint projection, force and viscosity are reconstructed from the impulse and Hodge ledgers, and the compatible commutation theorem gives

$$P_M^u \pi_m E_{\text{fin},M}^{k,\lambda_k}(Y_k) = E_{\text{proj-aff},M}^{k,\lambda_k}(\Pi_{M,k}(Y_k)), \quad R_{M,k} = 0;$$

and

- (h) the affine observation parameter t is not primitive: its event endpoints are the wake-realization values $\tau_k = \Theta_w(X_{\leq k})$, and laboratory timestamps arise only from the coincidence calibration (57.10) between the fluid history and a second finite reference-clock history.

Proof. The intrinsic refinement carrier and its Hausdorff metric follow from the admissible refinement system. The canonical transverse exchange follows from the carrier realization theorem. The complete retained-state estimate is the Wallis–Gregory bound in the transported norm; Theorem 55.6 supplies the stated sharper estimate for inherited orthogonal residual projections. The finite circulation estimate follows by bounding each canonical edge readout by its full retained carrier state, with initial bound S_0 including the inherited residuals. The vortex ancestry formula is finite substitution of the exact wake recurrence, while the ordered triadic cover rewrites each K_j as the history-conditioned output of its local three-body closure.

For the fluid readout, the wake-realization map fixes $\Delta\tau_k$ before any continuum coordinate is introduced. At event k , the incoming wake state a_k^w first deposits $r_k = D_{M,k}(a_k^w)$, which enters $q_k = B_M(m_k, m_k) + r_k$ and therefore the k -th momentum update. The same incoming wake state is then propagated by $a_{k+1}^w = M_k^w a_k^w$, furnishing $r_{k+1} = D_{M,k+1}(a_{k+1}^w)$ for the subsequent event.

Thus the retained wake is dynamically inherited rather than externally supplied, with an explicit left-endpoint causal order. The pressure Poisson problem is solvable on the connected periodic complex after removing the constant harmonic mode, and it preserves $m_{k+1} \in \ker d_2$. Whitney commutation proves exact divergence freedom. Applying P_M^u to the one-event law gives the endpoint identity (57.70); affine interpolation of those already completed endpoints gives (57.75) almost everywhere on the event interval. Compatibility of B_M , tensor divergence, pressure gradient, Hodge viscosity, and forcing with their reconstructions proves the commuting diagram and hence $R_{M,k} = 0$. No step treats the continuum field or its parameter t as the primary physical state.

Corollary 66.2 (Wall-bounded projected-affine balance). Replace the periodic geometry in the preceding theorem by an orientable wall complex satisfying every hypothesis of Theorem 58.1. Then the finite carrier, wake, ancestry, and Wallis–Gregory conclusions are unchanged, while the fluid readout conclusions hold with the boundary operators and traction term of (58.1)–(58.5). Theorem 58.3 supplies the required boundary complex, Hodge matrices, trace maps, and pressure compatibility for impermeable walls. Other wall conditions retain their explicitly stated hypotheses. Residual re-entry is counted through the live-channel balance on the actual wall wiring.

Corollary 66.3 (Constructed classical fluid projection). Proposition 60.4 gives a global smooth Navier–Stokes field by circular vector projection, with an exact finite event law and retained stress $\tau_{cl} = hI$. Its classical pressure is $p^\sharp = p + h$. For any compatible periodic readout in Definition 60.2, Proposition 60.3 characterizes this pressure absorption exactly by $\mathbb{P}\nabla \cdot \tau_{cl} = 0$. The Lamb decomposition, Kelvin circulation law, vorticity ancestry formula, and Jacobi determinant identity apply to these classical fields.

Corollary 66.4 (Constructed successive-wake realization). The completed aperture (54.8), transported by (56.1)–(56.3), gives a finite handoff with exact remnant return, exact ancestral memory, and preservation of the complete normalized norm. For its Wallis–Gregory recurrence with packets of norm at most E_w ,

$$\sup_N \|x_N\|_N \leq \frac{3\pi}{8} \|x_0\|_0 + \frac{E_w \pi}{96}.$$

For the co-transported packet of Theorem 55.5, the packet bound is also proved from its initial value and the sharper exact identity (55.15) applies. For the inherited orthogonal-projection recurrence of Theorem 55.6, the complete-state bound is $\sup_N \|x_N\|_N \leq (73\pi/192)\|x_0\|_0$. Its canonical carrier readouts obey (56.11), with S_0 measuring the complete initial wake.

For occupied histories with the anticonformal source clock (57.3), the same finite event construction has $\tau_N \geq \tau_0 + N(1 - \eta)\sigma_*$. Thus its nonaccumulation follows from that source law, with the history-conditioned jitter retained. The cochain momentum balance (57.45) is unchanged.

Proof. Theorem 54.2 proves the completed norm identity, Proposition 56.1 transports it, and Theorem 52.4 proves the finite inheritance formula. Corollary 55.4 supplies the packet estimate, and Theorem 55.6 gives the inherited-projection estimate. The canonical readout bound and Cauchy–Schwarz give (56.11). Theorem 57.1 proves nonaccumulation from the elementary source durations; Theorem 57.11 gives global continuation for the original finite momentum–wake law. Every finite update is defined by the displayed finite maps. $\square$

Theorems 57.19 and 58.3 construct the periodic and impermeable-wall spatial operators in the native balance. A specified linear momentum observation $\mathcal{C}_k$ transfers the retained-state estimate by the operator-norm inequality in Remark 57.21. This gives the exact map and norm to evaluate for each occupied realization.

67. Native dynamics and observational representations

The native fluid state consists of the momentum cochain together with the complete retained wake and its occupied event data. The source clock fixes $\tau_k = \Theta_w(X_{\leq k})$. Theorems 57.9 and 57.11 construct its successor at every event and continue that history through every finite source-clock interval at fixed resolution. The pressure constraint, wake deposition, and viscosity are finite operators in this construction.

The observational statements are obtained by applying specified maps to that history:

$$\begin{array}{ll}
 \text{native event and source clock} & \text{global continuation at fixed } M, \\
 \text{compatible Whitney observation} & \text{exact projected affine balance,} \\
 \text{compatible event commutation} & R_{M,k} = 0, \\
 \text{constructed circular vector projection} & \text{global smooth finite-mode classical fluid.}
 \end{array} \tag{67.1}$$

The logical direction is from the retained event to its observation. Neither the native successor nor its continuation requires an ambient smooth trajectory or a refinement limit. Laboratory timestamps are obtained by synchronizing two finite histories through Definition 57.2.

For an occupied pair of native states, let $\mathcal{O}_{M,k}$ denote the endpoint projection followed by the affine interpolation (57.73). The dependencies can then be written as

$$\begin{aligned}
 Y_k &\xrightarrow{E_{\text{fin},M}^{k,\lambda_k}} Y_{k+1}, & \tau_{k+1} &= \tau_k + \Delta\tau_k, \\
 (Y_k, Y_{k+1}) &\xrightarrow{\mathcal{O}_{M,k}} u_M|_{[\tau_k, \tau_{k+1}]}, \\
 X &\xrightarrow{P_{\text{cl}}} (u, p, f) \quad \text{under Definition 60.2.}
 \end{aligned} \tag{67.2}$$

Here X is the occupied history used by the smooth observation. The affine balance follows exactly from the completed event endpoints. The classical identification concerns the regularity and nonlinear readout of P_{cl} , together with the pressure-absorption condition of Proposition 60.3 when the retained stress is nonzero. These conditions belong to that representation.

Compatible observation has zero commutation defect by Theorem 57.16. For another observation $\tilde{P}_M$, equation (57.85) measures its defect $\tilde{R}_{M,k}$. Both observations act on the same retained history. A non-injective observation can preserve its projected law while omitting sign structure, wake ancestry, and refinement history, as expressed by the fiber criterion of Proposition 57.18.

Part VII

Master finite-support projection statement

68. Master finite-support projection theorem

Theorem 68.1 (Finite-support projection principle). Consider a finite refinement system with compact ultrametric branch boundary, persistent sign classes, an explicit carrier realization where needed, a finite event evolution E_k , and an observation D_k .

- (i) A sign carrier exists whenever persistent opposite sign classes coexist in a nested sequence of shrinking cylinders.
- (ii) Under regular b -fold refinement with contraction ρ , its intrinsic dimension is $\log b / (-\log \rho)$.
- (iii) The finite prime staircase has period variation $2\varphi(Q_q)$ and normalized variation 2. Its zero endpoint pair omits this transport information. Every prime belongs to the exhaustive ordered admission registry.
- (iv) An observation carries a unique closed successor map on its image exactly when

$$D_k(s) = D_k(t) \implies D_{k+1}(E_k(s)) = D_{k+1}(E_k(t)).$$

For linear events and observations this is equivalent to $E_k(\ker D_k) \subseteq \ker D_{k+1}$. For arbitrary finite deterministic histories, Theorem 38.5 constructs an observation extension satisfying the fiber condition.

- (v) Exact agreement of a compatible projected law with the finite event dynamics therefore does not imply that the projected object reconstructs the full finite history.

Proof. Items (i)–(iv) follow from the carrier-existence, regular-dimension, finite prime-variation, exhaustive-admission, and quotient-factorization theorems. Item (iv) uses the general fiber theorem 38.1, with its linear specialization proved in Section 7. Item (v) follows whenever an observation fiber contains distinct states: exact factorization controls their observed successors but cannot invert that fiber. $\square$

68.1. Constructive closure

The finite carrier, metric, observation, and cochain constructions give explicit objects for the hypotheses used in the projection principle. Theorem 3.4 constructs both persistent sign classes, Proposition 6.3 constructs the compatible metric, and Theorems 38.4 and 38.5 construct exact observation memory. The Whitney and wall constructions supply the stated pressure inverse, constraint-preserving operators, and nonzero closed-sector nonlinear flux.

The wake estimates include the full returning remnant for every passive aperture through Theorem 54.3. Corollary 54.4 derives passivity for orthogonal compressions of the transported isometry. Theorem 55.6 derives a uniform bound directly from the initial state when each remainder is read from that inherited state. Its orthogonal-projection constant is $73\pi/192$, with no separate packet-budget assumption and no common-direction requirement. The source clock and the finite event law give global native continuation at fixed resolution by Theorem 57.11. Complete-wake observation commutes by Corollary 57.17.

The explicit elliptic-output problem has both its lower and upper cost bounds in Theorem 39.4. The retained-deficit construction in Theorem 49.4 gives a finite deterministic frequency bound, and Corollary 49.5 derives its quadratic weights from the two-dimensional sector maps.

68.2. Arithmetic, computational, and physical realizations

Prime registry and stationary readout. Proposition 28.11 includes every prime in a finite admission prefix. The exact remainder handoff and the stationary Mellin value retain the arithmetic state. For a nontrivial zero $\rho = \sigma + it$,

$$K(\rho) = \frac{1}{\rho - 1}, \quad \mathcal{D}(\rho) = \frac{2\sigma - 1}{|\rho - 1|^2}.$$

Thus localization in the canonical coordinate is equivalent to $\mathcal{D}(\rho) = 0$. Proposition 29.11 gives the corresponding reflected work criterion. Prime completeness and this analytic localization criterion are separately specified mathematical statements.

Goldbach survivor count. The exact interval count of Theorem 32.3 is positive precisely when

$$\Delta_N > -\frac{(N-3)M_N}{Q_N} - 2e_N + \mathbf{1}_{\{N=4\}} \quad (N \geq 4 \text{ even}). \quad (68.1)$$

All quantities are determined by the finite survivor ledger.

Arithmetic and geometric representatives. A rational-point generator map is specified by independent points spanning $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Equality of their number with r_{an} is the rank criterion. A rational Hodge class represented by a finite rational combination of closed analytic codimension- k traces has an algebraic-cycle representative by Chow closure. The divisor and product-projective constructions give such representatives on their stated classes.

Unconditional computational lower bound. Theorem 39.4 proves

$$L_m = \Theta(4^m), \quad T_{\Pi}(m) = \Omega(4^m) \quad \text{for every correct } \Pi, \quad T_A(m) = O((m+1)4^m)$$

for the stipulated ordinary-binary elliptic output. The explicit certificate is the number of output symbols still owed. The decision-cost transfer theorem specifies a distinct interface through its quantified operation certificate.

Measurement realization. The planar sector maps and retained-deficit successor rule give quadratic occupation weights and a finite deterministic discrepancy bound. The event map determines the allocation law and its frequencies.

Inherited fluid transport. Successive wakes contain the transported remnants of all preceding events. The actual occupied-work identity gives the complete-state norm balance. The returning-aperture construction retains the live residual channels, and Theorem 55.6 gives

$$\sup_N \|x_N\|_N \leq \frac{73\pi}{192} \|x_0\|_0$$

for its inherited orthogonal-projection recurrence. The source clock gives global native continuation at fixed resolution by Theorem 57.11.

Circular vector projection. Lemma 60.1 proves smoothness directly from finitely many circular carriers and their vector projections. Proposition 60.4 constructs the global classical fluid family and its exact commuting event law. Nonzero retained isotropic stress enters the pressure channel. For compatible periodic readouts with fixed forcing, Proposition 60.3 gives the exact pressure-absorption criterion.

Compact-sector coercivity. The coercive estimate concerns the specified compact-sector operator and its norm. Any application to another Hamiltonian uses an explicit comparison of those operators and state spaces.

69. Conclusion

A finite ordered system may retain ancestry, variation, branch structure, or wake information that a scalar, lineal, or smooth readout does not reconstruct. The manuscript proves this distinction through exact finite variation, non-injective observation fibers, and retained event laws. Every prime occurs in the arithmetic registry, and every finite admission carries the retained remainder. Complete observation memory is constructed for finite states and fixed finite-dimensional linear event families. The native fluid event continues uniquely through every finite interval of the anticonformal source clock at fixed resolution, retaining inherited remnants and their signed work.

The computational results distinguish exact generation from explicit output: the shared certificate recurrence has a polynomial-cost generator, while cancellation-free elliptic doubling gives $L_m = \Theta(4^m)$, an output cost $\Omega(L_m)$, and a matching upper bound up to a logarithmic factor. The arithmetic and physical conclusions likewise follow their specified maps. Stationary prime states recover their canonical coordinate; the Goldbach ledger has an exact survivor count; wake-dependent measurement frequencies obey a concentration bound. The inherited-projection estimate derives a uniform wake bound from the initial state, and the retained-deficit event gives quadratic sector frequencies with a finite error bound. Circular vector projection proves smoothness for the constructed classical fluid family, and its inherited stress is retained through the pressure readout. Section [68.2](#) collects these constructions and their exact interfaces.

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