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Original Paper

Phenomenological Velocity and Bell–CHSH: Exceptional-Locus Semantics, Selection Simulations of $-\cos$, and a Microcausal Realization

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Abstract: Bell–CHSH is often read as a universal prohibition on *local hidden variables*. Mathematically, however, the CHSH bound $S \le 2$ is a theorem about a particular hypothesis class: a *single* Kolmogorov probability space carrying all counterfactual outcomes, together with measurement independence, Bell locality (factorized response functions), and bounded outputs. On that class the proof is pointwise and unconditional, so no algebraic reformulation of a parameter (including a "phenomenological velocity" (PV)) can yield Tsirelson-scale singlet correlations $E(a, b) = -\cos(a - b)$ on a Tsirelson quartet. We then make precise two distinct ways in which PV can remain meaningful as a *local* and in that sense "hidden" parameter while coexisting with Tsirelson-scale correlations, without contradicting Bell's theorem. First, we formalize "exceptional-locus/defined-only" conventions as setting-dependent acceptance (post-selection): reported correlators are conditional expectations under a setting-indexed family of accepted laws $\{\nu_{ab}\}$. Within this rung-1 semantics we give an explicit Bell-local deterministic base model whose accepted-sample joint law exactly matches the singlet law (unbiased marginals and correlator $-\cos(a - b)$) for all a, b, via an explicit acceptance rule $\gamma(a, b, \lambda) \in [0, 1]$. Second, we give a PV-indexed microcausal (operator-algebraic) realization: PV parameterizes a canonical $SU(2)$ unitary that conjugates local observables inside commuting Alice/Bob subalgebras, and in the singlet state the unconditional correlator equals $-\cos(a - b)$ and attains Tsirelson's value $2\sqrt{2}$. The results clarify a semantic and structural trichotomy: any "PV-to-Tsirelson" account must (i) change the effective ensemble by selection/context, or (ii) leave the Kolmogorov (commutative) model class (e.g. via noncommutativity or quasi-probability), or (iii) relax a Bell premise such as measurement independence or Bell locality. In this sense, Bell–CHSH constrains a specific formalization of hidden variables rather than ownership of the terms "local", "real", or "hidden" in physics discourse.

Keywords: Bell inequalities; CHSH; post-selection; unfair sampling; exceptional locus; total variation; microcausality; C*-algebras; Tsirelson bound

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1. Introduction

1.1. Bell–CHSH as a single-ensemble theorem Bell–CHSH [1,2] is a theorem about *unconditional* expectations taken under a *single* probability law: under measurement independence (a single prior law for the hidden variable), Bell locality (local response functions), and bounded outcomes, the CHSH value on any quartet satisfies $S \leq 2$. Because the inequality is proved pointwise in the hidden variable and then integrated under one measure, it cannot be evaded by algebraic reformulations *within* the same Kolmogorov hypothesis class.

1.2. Phenomenological velocity, exceptional loci, and semantic ambiguity A recurrent motif in PV-based discussions is that certain radical expressions become undefined (e.g. $0/0$), or become multivalued due to branch structure, and that one can adopt a semantics in which the exceptional locus is bracketed away, regularized, or resolved by convention. If such semantics influences which trials are used when reporting correlations, then it acts as a form of post-selection/conditioning. If, instead, the semantics produces bounded outputs for all trials and no conditioning occurs, then the resulting model remains in Bell's unconditional hypothesis class and obeys CHSH ≤ 2.

The purpose of this paper is to make this dichotomy (and its consequences) mathematically explicit.

1.3. Main results (informal) We prove:

(R1) **Unconditional no-go.** No Bell-local measurement-independent unconditional model can reproduce Tsirelson-scale $-\cos(a-b)$ on a Tsirelson quartet.

(R2) **Selection realization.** There exists a Bell-local measurement-independent deterministic base model and an explicit setting-dependent acceptance rule $\gamma(a, b, \lambda) \in [0, 1]$ such that the accepted-sample *joint law* has unbiased marginals and correlator $E_{\text{obs}}(a, b) = -\cos(a - b)$ for all a, b.

(R3) **Microcausal realization.** There exists a PV-indexed microcausal (operator-algebraic) model in which PV parameterizes local observables by unitary conjugation inside commuting Alice/Bob subalgebras, and in the singlet state the unconditional correlator equals $-\cos(a - b)$.

Operational relevance (fair sampling and coincidence logic). Rung-1 acceptance $\gamma(a, b, \lambda)$ is the measure-theoretic abstraction of standard laboratory pipelines: detector non-clicks, voltage thresholds, time-window coincidence pairing, invalid-event flags, and any "defined-only" exceptional-locus convention that discards trials on a setting-dependent locus. In such cases the reported correlator is a conditional expectation with respect to a setting-indexed accepted law ν_{ab}, and the usual CHSH proof step (integration under one common measure) is no longer applicable. This places "exceptional-locus semantics" in direct contact with the detection/coincidence loophole literature and with the fair-sampling assumption used when mapping raw logs to reported correlators.

Relation to superdeterminism (clarifying remark). Throughout, rung-1 selection explanations are formulated with measurement independence at emission (a fixed prior ρ); the setting dependence enters through conditioning on acceptance. This is logically distinct from superdeterministic models, in which the hidden-variable prior itself depends on the chosen settings.

1.4. Structure of the paper Section 2 fixes the model classes, the selection-as-acceptance formalism, and the needed metric geometry (total variation, dispersion). Section 3 performs the PV radical audit and proves the strict-semantics cancellation result. Section 4 gives the explicit $-\cos$ construction via setting-dependent acceptance and records the unconditional rung-0 no-go. Section 5 gives the PV-indexed microcausal realization and the full derivation of $-\cos$. Appendices (if included) may collect longer algebraic details and endpoint conventions.

2. Framework: Model Classes, Acceptance Semantics, and CHSH

2.1. Measurable structures and conventions We treat settings and hidden variables in a standard measure-theoretic way.

- The setting spaces S_A, S_B are assumed to be measurable spaces (in typical applications, standard Borel spaces).
- The hidden-variable space $(\Lambda, \mathcal{F})$ is a measurable space and $\rho \in \mathcal{P}(\Lambda)$ is a probability measure.
- All maps in (a, b, λ) that appear under integrals are assumed measurable with respect to the product σ-algebra.

Remark 2.1 (Total variation convention). We use $\mathrm{TV}(\mu, \nu) := \sup_E |\mu(E) - \nu(E)| \in [0, 1]$ for probability measures. Some authors define total variation with an additional factor of $1/2$; our bounds are consistent with the present convention. We adopt the $[0, 1]$ convention so that later dual bounds (e.g. $|\int f\,d\mu - \int f\,d\nu| \leq 2\,\mathrm{TV}(\mu, \nu)$ for $\|f\|_\infty \leq 1$) and the resulting CHSH-inflation inequalities do not carry extraneous $\frac{1}{2}$ factors.

2.2. Rung-0 (Bell/CHSH) hypothesis class: a single Kolmogorov ensemble

The classical Bell–CHSH hypothesis class is a *single-ensemble* class: all correlations are unconditional expectations under one probability law.

Definition 2.2 (Bell-local measurement-independent (MI) model; unconditional semantics)**.** Fix setting sets S_A, S_B. A *Bell-local measurement-independent model with unconditional semantics* consists of:

(i) a measurable space $(\Lambda, \mathcal{F})$ and a probability measure $\rho \in \mathcal{P}(\Lambda)$;

(ii) measurable response functions

$$A : S_A \times \Lambda \to [-1, 1], \qquad B : S_B \times \Lambda \to [-1, 1]; \tag{1}$$

(iii) *measurement independence (MI):* the prior law ρ is fixed and does not depend on the chosen settings;

(iv) *unconditional correlators:* for each $(a, b) \in S_A \times S_B$,

$$E_{\text{full}}(a, b) := \int_\Lambda A(a, \lambda)\, B(b, \lambda)\, d\rho(\lambda). \tag{2}$$

Remark 2.3 (Bounded outcomes). The $[-1, 1]$ bound is the natural normalization for CHSH. Deterministic $\{\pm 1\}$ models are included as a special case.

2.3. CHSH functional and the unconditional inequality

Fix a CHSH quartet of settings $a_0, a_1 \in S_A$ and $b_0, b_1 \in S_B$. Write

$$A_i(\lambda) := A(a_i, \lambda), \qquad B_j(\lambda) := B(b_j, \lambda), \qquad i, j \in \{0, 1\}. \tag{3}$$

Define the unconditional correlators $E_{ij}^{\text{full}} := E_{\text{full}}(a_i, b_j)$ and the unconditional CHSH value

$$S_{\text{full}} := \left|E_{00}^{\text{full}} + E_{01}^{\text{full}} + E_{10}^{\text{full}} - E_{11}^{\text{full}}\right|. \tag{4}$$

Lemma 2.4 (Pointwise CHSH algebra)**.** *For any real numbers* $A_0, A_1, B_0, B_1 \in [-1, 1]$,

$$\left|A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1\right| \leq 2. \tag{5}$$

Proof. Rewrite the expression as

$$A_0(B_0 + B_1) + A_1(B_0 - B_1). \tag{6}$$

By the triangle inequality and $|A_0|, |A_1| \leq 1$,

$$|A_0(B_0 + B_1) + A_1(B_0 - B_1)| \leq |B_0 + B_1| + |B_0 - B_1|. \tag{7}$$

For real x, y one has $|x + y| + |x - y| = 2\max\{|x|, |y|\}$, hence

$$|B_0 + B_1| + |B_0 - B_1| \leq 2\max\{|B_0|, |B_1|\} \leq 2. \tag{8}$$

Theorem 2.5 (Unconditional Bell–CHSH)**.** *Every model in Definition 2.2 satisfies* $S_{\text{full}} \leq 2$ *on every CHSH quartet.*

Proof. Define the measurable function

$$C(\lambda) := A_0(\lambda)B_0(\lambda) + A_0(\lambda)B_1(\lambda) + A_1(\lambda)B_0(\lambda) - A_1(\lambda)B_1(\lambda). \tag{9}$$

Then, by linearity of the integral,

$$\int_\Lambda C(\lambda)\, d\rho(\lambda) = E_{00}^{\text{full}} + E_{01}^{\text{full}} + E_{10}^{\text{full}} - E_{11}^{\text{full}}. \tag{10}$$

Hence

$$S_{\text{full}} = \left|\int_\Lambda C\, d\rho\right| \leq \int_\Lambda |C|\, d\rho \leq \int_\Lambda 2\, d\rho = 2, \tag{11}$$

where the second inequality uses Lemma 2.4 pointwise in λ.

Remark 2.6 (Immediate no-go consequence). If a target correlator table on a quartet yields a CHSH value strictly larger than 2 (e.g. the Tsirelson $-\cos$ table yields $2\sqrt{2}$ on a standard quartet), then that table cannot arise from unconditional expectations in the rung-0 hypothesis class.

2.4. Rung-1 semantics: setting-dependent acceptance (selection as conditionalization)

In practice, reported correlations may be computed after discarding trials via coincidence logic, time windows, thresholds, invalid flags, or "defined-only" exceptional-locus conventions. Mathematically, this is a conditioning mechanism (cf. the detection/coincidence loophole literature [3–7]).

Definition 2.7 (Acceptance rule)**.** An *acceptance rule* is a measurable function

$$\gamma : S_A \times S_B \times \Lambda \to [0,1], \qquad (a,b,\lambda) \mapsto \gamma(a,b,\lambda), \tag{12}$$

interpreted as the conditional probability that a trial is accepted given (a,b,λ).

Remark 2.8 (Randomized acceptance entails no loss of generality). If the pipeline acceptance decision uses additional randomness beyond λ, one may enlarge Λ to $\Lambda \times [0,1]$ and absorb that randomness into λ. Thus, modeling acceptance as a measurable $\gamma \in [0,1]$ is without loss of generality.

Definition 2.9 (Acceptance rate and accepted law)**.** Assume

$$Z(a,b) := \int_\Lambda \gamma(a,b,\lambda)\, d\rho(\lambda) \in (0,1] \qquad (\forall (a,b) \in S_A \times S_B), \tag{13}$$

so the event "accepted" has strictly positive probability at every setting pair. Define the *accepted hidden-variable law* $\nu_{ab} \in \mathcal{P}(\Lambda)$ by

$$\nu_{ab}(E) := \frac{\int_E \gamma(a,b,\lambda)\, d\rho(\lambda)}{\int_\Lambda \gamma(a,b,\lambda)\, d\rho(\lambda)} = \frac{1}{Z(a,b)} \int_E \gamma(a,b,\lambda)\, d\rho(\lambda), \qquad E \in \mathcal{F}. \tag{14}$$

Remark 2.10 (Radon–Nikodym derivative). By construction $\nu_{ab} \ll \rho$ and

$$\frac{d\nu_{ab}}{d\rho}(\lambda) = \frac{\gamma(a,b,\lambda)}{Z(a,b)} \qquad (\rho\text{-a.e.}). \tag{15}$$

This is the "weight by acceptance and renormalize" rule.

Definition 2.11 (Accepted-sample (observed) correlator)**.** The correlator computed on accepted trials is

$$E_{\text{obs}}(a,b) := \int_\Lambda A(a,\lambda)B(b,\lambda)\, d\nu_{ab}(\lambda). \tag{16}$$

Equivalently,

$$E_{\text{obs}}(a,b) = \frac{1}{Z(a,b)} \int_\Lambda A(a,\lambda)B(b,\lambda)\, \gamma(a,b,\lambda)\, d\rho(\lambda). \tag{17}$$

Fix a quartet a_0, a_1, b_0, b_1 and abbreviate

$$\nu_{ij} := \nu_{a_i b_j}, \qquad E_{ij} := E_{\text{obs}}(a_i, b_j), \qquad i,j \in \{0,1\}. \tag{18}$$

The *accepted-sample* CHSH value is

$$S_{\text{obs}} := \left| E_{00} + E_{01} + E_{10} - E_{11} \right|. \tag{19}$$

Remark 2.12 (Where CHSH can inflate). In the unconditional rung-0 model, the same ρ is used for all four expectations in CHSH. Under acceptance, each E_{ij} is an expectation under the (possibly different) law ν_{ij}. If $\nu_{00}, \nu_{01}, \nu_{10}, \nu_{11}$ are not all equal, the usual CHSH integration step (no longer a single common measure) fails, and S_{obs} can exceed 2 even when A, B are Bell-local and ρ is measurement-independent.

2.5. Total variation, dispersion, and a universal CHSH inflation bound

Definition 2.13 (Total variation distance)**.** For $\mu, \nu \in \mathcal{P}(\Lambda)$,

$$\mathrm{TV}(\mu, \nu) := \sup_{E \in \mathcal{F}} |\mu(E) - \nu(E)|. \tag{20}$$

If Λ is finite, $\mathrm{TV}(\mu, \nu) = \frac{1}{2} \sum_{x \in \Lambda} |\mu(x) - \nu(x)|$.

Lemma 2.14 (TV controls bounded expectation errors)**.** *Let $\mu, \nu \in \mathcal{P}(\Lambda)$ and let $f : \Lambda \to \mathbb{R}$ be measurable with $\|f\|_\infty \leq 1$. Then*

$$\left| \int f \, d\mu - \int f \, d\nu \right| \leq 2 \, \mathrm{TV}(\mu, \nu). \tag{21}$$

Proof. A standard dual characterization of total variation is

$$\mathrm{TV}(\mu, \nu) = \frac{1}{2} \sup_{\|g\|_\infty \leq 1} \left| \int g \, d\mu - \int g \, d\nu \right|. \tag{22}$$

Apply this with $g = f$.

Definition 2.15 (Quartet dispersion and diameter)**.** Fix a CHSH quartet and let $\nu_{00}, \nu_{01}, \nu_{10}, \nu_{11}$ be the accepted laws on that quartet. Define the *quartet dispersion*

$$\Delta_Q := \inf_{\mu \in \mathcal{P}(\Lambda)} \Big(\mathrm{TV}(\nu_{00}, \mu) + \mathrm{TV}(\nu_{01}, \mu) + \mathrm{TV}(\nu_{10}, \mu) + \mathrm{TV}(\nu_{11}, \mu) \Big), \tag{23}$$

and the *quartet diameter*

$$D_Q := \max_{(i,j) \neq (k,\ell)} \mathrm{TV}(\nu_{ij}, \nu_{k\ell}). \tag{24}$$

Proposition 2.16 (Dispersion vanishes iff the accepted law is setting-independent on the quartet)**.** $\Delta_Q = 0$ *if and only if* $\nu_{00} = \nu_{01} = \nu_{10} = \nu_{11}$.

Proof. If all four measures coincide, choose $\mu = \nu_{00}$ in (23) and obtain $\Delta_Q = 0$.

Conversely, assume $\Delta_Q = 0$. Then there exists a sequence $\mu_n \in \mathcal{P}(\Lambda)$ with

$$\mathrm{TV}(\nu_{00}, \mu_n) + \mathrm{TV}(\nu_{01}, \mu_n) + \mathrm{TV}(\nu_{10}, \mu_n) + \mathrm{TV}(\nu_{11}, \mu_n) \to 0. \tag{25}$$

Fix any two indices (i, j) and (k, ℓ). By the triangle inequality for TV,

$$\mathrm{TV}(\nu_{ij}, \nu_{k\ell}) \leq \mathrm{TV}(\nu_{ij}, \mu_n) + \mathrm{TV}(\mu_n, \nu_{k\ell}). \tag{26}$$

The right-hand side tends to 0 as $n \to \infty$, hence $\mathrm{TV}(\nu_{ij}, \nu_{k\ell}) = 0$ and thus $\nu_{ij} = \nu_{k\ell}$. So all four measures are equal.

Proposition 2.17 (Elementary relations between Δ_Q and D_Q)**.** *For every quartet,*

$$D_Q \le \Delta_Q \le 3D_Q. \tag{27}$$

Proof. Lower bound $D_Q \le \Delta_Q$. Fix any $\mu \in \mathcal{P}(\Lambda)$. Choose indices $(i,j) \neq (k,\ell)$ attaining the diameter, so that $D_Q = \mathrm{TV}(\nu_{ij}, \nu_{k\ell})$. Then by the triangle inequality,

$$D_Q = \mathrm{TV}(\nu_{ij}, \nu_{k\ell}) \le \mathrm{TV}(\nu_{ij}, \mu) + \mathrm{TV}(\mu, \nu_{k\ell}) \le \sum_{r \in \{00,01,10,11\}} \mathrm{TV}(\nu_r, \mu). \tag{28}$$

Taking the infimum over μ yields $D_Q \le \Delta_Q$.

Upper bound $\Delta_Q \le 3D_Q$. Choose $\mu = \nu_{00}$ in (23). Then

$$\Delta_Q \le \mathrm{TV}(\nu_{01}, \nu_{00}) + \mathrm{TV}(\nu_{10}, \nu_{00}) + \mathrm{TV}(\nu_{11}, \nu_{00}) \le 3D_Q, \tag{29}$$

since each term is bounded by the maximum pairwise distance D_Q.

Remark 2.18 (Why the factor 3 appears, and tightness). The factor 3 is the trivial "$4-1$" bound: on a CHSH quartet there are four accepted laws, and if the reference law is taken to be one of them (e.g. $\mu = \nu_{00}$), exactly three distances remain, each bounded by the diameter D_Q. The constant 3 can be tight. For example, on $\Lambda = \{0,1,2,3\}$ let $\nu_{00} = \delta_0$, $\nu_{01} = \delta_1$, $\nu_{10} = \delta_2$, $\nu_{11} = \delta_3$. Then $D_Q = 1$, and for any probability measure μ one has $\mathrm{TV}(\delta_k, \mu) = 1 - \mu(\{k\})$, hence

$$\sum_{ij} \mathrm{TV}(\nu_{ij}, \mu) = \sum_{k=0}^{3} (1 - \mu(\{k\})) = 3,$$

so $\Delta_Q = 3D_Q$.

Theorem 2.19 (Universal CHSH inflation bound (reference-measure form))**.** *Assume Bell locality, measurement independence at emission (a fixed prior ρ), and bounded outcomes. Fix a quartet and write ν_{ij} for the accepted laws on that quartet. Then for every reference measure $\mu \in \mathcal{P}(\Lambda)$,*

$$S_{\mathrm{obs}} \le 2 + 2\Big(\mathrm{TV}(\nu_{00}, \mu) + \mathrm{TV}(\nu_{01}, \mu) + \mathrm{TV}(\nu_{10}, \mu) + \mathrm{TV}(\nu_{11}, \mu)\Big). \tag{30}$$

Proof. For each $(i,j) \in \{0,1\}^2$ define

$$f_{ij}(\lambda) := A(a_i, \lambda) B(b_j, \lambda), \tag{31}$$

so $\|f_{ij}\|_\infty \le 1$. Define

$$E_{ij} := \int f_{ij}\, d\nu_{ij}, \qquad \widetilde{E}_{ij} := \int f_{ij}\, d\mu. \tag{32}$$

Then

$$
\begin{aligned}
S_{\text{obs}} &= \big|E_{00} + E_{01} + E_{10} - E_{11}\big| \\
&= \Big|(\widetilde{E}_{00} + \widetilde{E}_{01} + \widetilde{E}_{10} - \widetilde{E}_{11}) + \big[(E_{00} - \widetilde{E}_{00}) + (E_{01} - \widetilde{E}_{01}) + (E_{10} - \widetilde{E}_{10}) - (E_{11} - \widetilde{E}_{11})\big]\Big| \\
&\leq \big|\widetilde{E}_{00} + \widetilde{E}_{01} + \widetilde{E}_{10} - \widetilde{E}_{11}\big| + \sum_{i,j\in\{0,1\}} |E_{ij} - \widetilde{E}_{ij}|. \qquad (33)
\end{aligned}
$$

Step 1 (single-measure CHSH). Since all $\widetilde{E}_{ij}$ are expectations under the *single* measure μ, Theorem 2.5 (with ρ replaced by μ) yields

$$
\big|\widetilde{E}_{00} + \widetilde{E}_{01} + \widetilde{E}_{10} - \widetilde{E}_{11}\big| \leq 2. \qquad (34)
$$

Step 2 (TV controls each error term). By Lemma 2.14,

$$
|E_{ij} - \widetilde{E}_{ij}| = \left|\int f_{ij}\, d\nu_{ij} - \int f_{ij}\, d\mu\right| \leq 2\,\mathrm{TV}(\nu_{ij}, \mu). \qquad (35)
$$

Substitute these bounds into (33) to obtain (30).

Corollary 2.20 (Intrinsic dispersion bound)**.** *On any quartet,*

$$
S_{\text{obs}} \leq 2 + 2\Delta_Q. \qquad (36)
$$

Proof. Take the infimum of (30) over $\mu \in \mathcal{P}(\Lambda)$ and use Definition 2.15.

Corollary 2.21 (Diameter bound)**.** *On any quartet,*

$$
S_{\text{obs}} \leq \min\{4,\ 2 + 6D_Q\}. \qquad (37)
$$

Proof. Choose $\mu = \nu_{00}$ in (30). Then

$$
S_{\text{obs}} \leq 2 + 2\big(0 + \mathrm{TV}(\nu_{01}, \nu_{00}) + \mathrm{TV}(\nu_{10}, \nu_{00}) + \mathrm{TV}(\nu_{11}, \nu_{00})\big) \leq 2 + 6D_Q. \qquad (38)
$$

Also, since each $E_{ij} \in [-1, 1]$, trivially $S_{\text{obs}} \leq 4$. Take the minimum.

Corollary 2.22 (Tsirelson-scale necessary dispersion in any Bell-local MI selection explanation)**.** *If a Bell-local MI model reproduces $S_{\text{obs}} = 2\sqrt{2}$ on a quartet* via setting-dependent acceptance, *then*

$$
\Delta_Q \geq \sqrt{2} - 1 \approx 0.4142, \qquad D_Q \geq \frac{\sqrt{2} - 1}{3} \approx 0.1381. \qquad (39)
$$

Proof. From Corollary 2.20, $2\sqrt{2} \leq 2 + 2\Delta_Q$ implies $\Delta_Q \geq \sqrt{2} - 1$. From Corollary 2.21, $2\sqrt{2} \leq 2 + 6D_Q$ implies $D_Q \geq (\sqrt{2} - 1)/3$.

3. Phenomenological Velocity Radicals and Exceptional-Locus Semantics

3.1. Principal square root convention To avoid ambiguity about branch choices, we fix the principal square root on $\mathbb{C}$. For $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in (-\pi, \pi]$, define

$$\sqrt{z} := \sqrt{r}\, e^{i\theta/2}. \tag{40}$$

On $\mathbb{R}_{\geq 0}$ this coincides with the usual nonnegative square root. When we speak of "strict semantics" below, we mean that each radical is interpreted as this single-valued function on its domain.

Lemma 3.1 (Scaling of the principal root by positive reals)**.** *Let $c > 0$ be real. Then for every $z \in \mathbb{C}$,*

$$\sqrt{c^2 z} = c\,\sqrt{z}. \tag{41}$$

Proof. Write $z = re^{i\theta}$ with $\theta \in (-\pi, \pi]$. Since $c^2 > 0$ has argument 0,

$$c^2 z = (c^2 r)e^{i\theta}, \tag{42}$$

hence

$$\sqrt{c^2 z} = \sqrt{c^2 r}\, e^{i\theta/2} = c\sqrt{r}\, e^{i\theta/2} = c\sqrt{z}. \tag{43}$$

3.2. PV radical architecture and the exceptional locus

We consider the PV radical expression in the form typically manipulated in computer algebra systems (CAS). Let $c > 0$ be fixed and let

$$\kappa := \sqrt{1 - \frac{v^2}{c^2}} \tag{44}$$

denote the (principal) square root, with $\kappa \in \mathbb{R}$ only when $|v| \leq c$.

To avoid notational collisions with the hidden-variable symbol λ and the acceptance map γ used elsewhere in the paper, we denote the PV algebraic auxiliaries by $L_\pm$ and the corresponding coefficients by g, τ. Introduce the auxiliary expressions

$$L_+ := l\alpha + xg - r\tau, \qquad L_- := l\alpha - xg + r\tau, \tag{45}$$

and consider the radical equation

$$\alpha l \sin\beta = \sqrt{L_+\kappa}\,\sqrt{L_-/\kappa}. \tag{46}$$

Define

$$D := (xg - r\tau)^2 - l^2\alpha^2\cos^2\beta. \tag{47}$$

Remark 3.2 (Exceptional locus). Expression (46) is generically *partial*: even if all symbols are real, the radicals may fail to be real-valued when their radicands are negative, and the product is undefined when $\kappa = 0$ (i.e. $v = \pm c$), because the quantity L_-/κ is not defined (including the $0/0$ case when $L_- = 0$). Such undefinedness defines an *exceptional locus* in parameter space. The Bell–CHSH relevance hinges on whether such undefinedness changes which trials contribute to reported averages (i.e. whether it induces selection).

3.3. Strict semantics: the radical equation forces $D = 0$ and does not determine v

Proposition 3.3 (Strict consequence of the PV radical equation)**.** *Assume* (46) *holds in a commutative scalar field (e.g.* $\mathbb{R}$ *or* $\mathbb{C}$*), and assume moreover that* $\kappa \neq 0$ *(equivalently* $v \neq \pm c$*) so that the division* L_-/κ *is defined, in a semantics in which:*

(i) $\sqrt{\cdot}$ *is interpreted as a single-valued function on its domain (the principal root);*

(ii) each occurrence of $\sqrt{z}$ *lies in its domain, so that* $(\sqrt{z})^2 = z$ *is valid;*

(iii) products are evaluated in the usual associative, commutative way.

Then (46) *implies* $D = 0$*, i.e.*

$$(xg - r\tau)^2 = l^2\alpha^2\cos^2\beta. \tag{48}$$

In particular, the squared compatibility constraint extracted from (46) *contains no* v*.*

Proof. Square both sides of (46). Under the stated hypotheses,

$$\begin{aligned}(\alpha l \sin\beta)^2 &= \left(\sqrt{L_+\kappa}\,\sqrt{L_-/\kappa}\right)^2 = \left(\sqrt{L_+\kappa}\right)^2\left(\sqrt{L_-/\kappa}\right)^2 \\ &= (L_+\kappa)\,(L_-/\kappa) = L_+L_- \qquad (\text{since } \kappa \neq 0).\end{aligned} \tag{49}$$

Now expand L_+L_- using (45):

$$L_+L_- = (l\alpha + xg - r\tau)(l\alpha - xg + r\tau) = (l\alpha)^2 - (xg - r\tau)^2. \tag{50}$$

Therefore

$$(l\alpha)^2\sin^2\beta = (l\alpha)^2 - (xg - r\tau)^2 \quad \Longrightarrow \quad (xg - r\tau)^2 = (l\alpha)^2\cos^2\beta, \tag{51}$$

which is exactly $D = 0$ by (47). No term in this derivation involves v (equivalently κ), so the resulting constraint is v-free.

Remark 3.4 (Interpretation). Proposition 3.3 is purely algebraic: in strict semantics, the PV-dependent factor κ cancels after squaring, so the radical equation does not constrain v. Thus, within an unconditional Bell-local MI model, (46) cannot by itself define a nontrivial hidden variable v whose distribution would then generate Tsirelson-scale correlations by ordinary averaging.

3.4. The CAS ratio $v = \pm\sqrt{c^2 D}/\sqrt{D}$: collapse off-locus and undefinedness on-locus

Proposition 3.5 (Collapse and exceptional-locus undefinedness of the CAS ratio)**.** *Assume $c > 0$ is real and $\sqrt{\cdot}$ denotes the principal square root. Define*

$$v_{\mathrm{CAS}}(D) := \pm\frac{\sqrt{c^2 D}}{\sqrt{D}}. \tag{52}$$

Then:

(i) for every $D \neq 0$ in the domain of $\sqrt{\cdot}$, one has $v_{\mathrm{CAS}}(D) = \pm c$;

(ii) at $D = 0$, the expression is of indeterminate form $0/0$ and is undefined as an ordinary number.

Consequently, v_{CAS} cannot parameterize a nontrivial continuum of values on the strict solution locus of (46)*, because that locus forces $D = 0$ by Proposition 3.3.*

Proof. By Lemma 3.1, $\sqrt{c^2 D} = c\sqrt{D}$ for every D in the principal-root domain. Hence for $D \neq 0$,

$$v_{\mathrm{CAS}}(D) = \pm\frac{c\sqrt{D}}{\sqrt{D}} = \pm c. \tag{53}$$

At $D = 0$, both numerator and denominator are 0, so the ratio is undefined as a number. Finally, Proposition 3.3 shows that strict solutions of (46) satisfy $D = 0$, so v_{CAS} is undefined on the strict solution locus.

3.5. Exceptional-locus semantics as acceptance: "defined-only" ⇒ selection

The Bell–CHSH relevance is measure-theoretic: if undefinedness/branch handling changes which trials contribute to the reported correlator, then the reported correlator is a *conditional* expectation under an accepted law ν_{ab}, i.e. a rung-1 object.

Definition 3.6 (Partial outputs and the definedness/acceptance indicator)**.** Let $A^\star : S_A \times \Lambda \to \{\pm 1\} \cup \{\perp\}$ and $B^\star : S_B \times \Lambda \to \{\pm 1\} \cup \{\perp\}$ be *partial* response rules, where $\perp$ denotes "undefined". Define the *definedness/acceptance indicator*

$$\mathrm{Acc}(a, b, \lambda) := \mathbf{1}\{A^\star(a, \lambda) \neq \perp\}\,\mathbf{1}\{B^\star(b, \lambda) \neq \perp\} \in \{0, 1\}. \tag{54}$$

Proposition 3.7 (Defined-only semantics is selection/conditioning)**.** *Fix an MI prior ρ on Λ. Suppose a pipeline reports the correlation at settings (a, b) by computing the product $A^\star(a, \lambda)B^\star(b, \lambda)$* only on trials where it is defined*, i.e. only on $\{\mathrm{Acc}(a, b, \lambda) = 1\}$. Assume $\rho(\mathrm{Acc}(a, b, \cdot) = 1) > 0$ for the settings under discussion. Define $\gamma(a, b, \lambda) := \mathrm{Acc}(a, b, \lambda)$ and define the accepted law ν_{ab} by* (14)*. Then the reported correlator equals*

$$\mathbb{E}_\rho\big[A^\star(a, \lambda)B^\star(b, \lambda) \mid \mathrm{Acc}(a, b, \lambda) = 1\big] = \int_\Lambda A(a, \lambda)B(b, \lambda)\, d\nu_{ab}(\lambda), \tag{55}$$

for any bounded completion A, B of $A^\star, B^\star$ on the rejected locus $\{\mathrm{Acc} = 0\}$.

Proof. By definition of conditional expectation on an event of positive probability,

$$\mathbb{E}_\rho\big[A^\star(a,\lambda)B^\star(b,\lambda) \mid \mathrm{Acc}(a,b,\lambda)=1\big] = \frac{\int_\Lambda A^\star(a,\lambda)B^\star(b,\lambda)\,\mathrm{Acc}(a,b,\lambda)\,d\rho(\lambda)}{\int_\Lambda \mathrm{Acc}(a,b,\lambda)\,d\rho(\lambda)}. \tag{56}$$

On the accepted locus $\{\mathrm{Acc} = 1\}$ we may replace $A^\star, B^\star$ by any completion $A, B \in [-1,1]$ because Acc zeroes out the rejected locus. Thus the numerator equals $\int A(a,\lambda)B(b,\lambda)\gamma(a,b,\lambda)\,d\rho(\lambda)$ and the denominator equals $Z(a,b)$. By Definition 2.9, this ratio is exactly $\int A(a,\lambda)B(b,\lambda)\,d\nu_{ab}(\lambda) = E_{\mathrm{obs}}(a,b)$.

Remark 3.8 (Bell relevance). Proposition 3.7 is the formal bridge from exceptional-locus semantics to the acceptance formalism: any defined-only convention *is* a selection map γ. If γ is setting dependent (hence ν_{ab} varies with (a,b)), then S_{obs} is governed by the inflation bounds of Section 2.5.

3.6. A trichotomy for PV-to-Tsirelson claims

Theorem 3.9 (PV-to-Tsirelson trichotomy)**.** *Assume a model uses a single MI prior at emission and Bell-local response rules. Suppose the model purports to reproduce Tsirelson-scale correlations (e.g. $S = 2\sqrt{2}$ on some CHSH quartet). Then at least one of the following must occur:*

(i) ***Rung-1 ensemble shift (selection/context).*** *The reported correlators are conditional expectations under a setting-indexed family of accepted laws $\{\nu_{ab}\}$. In that case, S_{obs} is bounded by $2 + 2\Delta_Q$ (Corollary 2.20), and Tsirelson-scale values require $\Delta_Q \geq \sqrt{2} - 1$ (Corollary 2.22).*

(ii) ***Exit from the Kolmogorov (commutative) hypothesis class.*** *The model uses noncommutative (operator-algebraic) probability, signed/quasi-probabilities, or another non-Kolmogorov state notion. In that case Tsirelson $2\sqrt{2}$ can occur unconditionally in microcausal models (Section 5).*

(iii) ***Relaxation of a Bell hypothesis.*** *Measurement independence or Bell locality (in the classical sense) fails, explicitly or implicitly.*

Proof. If neither (ii) nor (iii) occurs and the model also avoids (i), then all reported correlators are unconditional expectations under a single Kolmogorov measure ρ with Bell-local bounded response functions, i.e. the model lies in Definition 2.2. Then Theorem 2.5 forces $S \leq 2$ on every quartet, contradicting Tsirelson-scale $S = 2\sqrt{2}$. Therefore at least one of (i)–(iii) must occur.

4. Selection Simulation of the Singlet Law: An Explicit Bell-Local Deterministic Base Model with $E_{\text{obs}}(a,b) = -\cos(a-b)$

4.1. Settings, target correlation, and a canonical Bell-local deterministic base model

Throughout this section we take settings to be planar angles on the circle:

$$S_A = S_B = \mathbb{R}/2\pi\mathbb{Z}. \tag{57}$$

For $a, b \in \mathbb{R}/2\pi\mathbb{Z}$, define the *principal angular separation*

$$\Delta(a,b) := \arccos(\cos(a-b)) \in [0,\pi]. \tag{58}$$

Equivalently, if a, b are represented in $[0, 2\pi)$, then $\Delta(a,b) = \min\{|a-b|,\, 2\pi - |a-b|\}$.

The singlet target correlation is

$$E_{\text{tgt}}(a,b) := -\cos(a-b) = -\cos\big(\Delta(a,b)\big). \tag{59}$$

Hidden-variable space and MI prior. Let

$$\Lambda_0 := [0, 2\pi) \quad \text{with Lebesgue } \sigma\text{-algebra, and} \quad \varphi \sim \text{Unif}([0,2\pi)). \tag{60}$$

Write ρ_0 for the uniform law of φ. This prior is measurement-independent (MI) by construction.

Deterministic local outputs. Define the sign convention

$$\text{sgn}(x) := \begin{cases} +1, & x \geq 0, \\ -1, & x < 0. \end{cases} \tag{61}$$

(Any choice at $x = 0$ yields the same correlators because the zero-set has ρ_0-measure 0.)

Define deterministic Bell-local responses

$$A(a,\varphi) := \text{sgn}(\cos(\varphi - a)), \qquad B(b,\varphi) := -\,\text{sgn}(\cos(\varphi - b)). \tag{62}$$

Then $A, B \in \{\pm 1\}$ and are measurable.

Define the product

$$f_{ab}(\varphi) := A(a,\varphi)\, B(b,\varphi) \in \{\pm 1\}. \tag{63}$$

Remark 4.1 (Why this base model is relevant). Model (62) is a canonical deterministic Bell-local MI model for angle settings. Its unconditional correlation is *not* $-\cos(a-b)$ but rather a triangular wave (next subsection). This makes it an ideal base model for illustrating how setting-dependent acceptance can convert an unconditional rung-0 correlation into a Tsirelson-scale accepted-sample correlation.

4.2. Unconditional correlator of the sign–cos base model (triangular wave)

Proposition 4.2 (Unconditional correlation of (62))**.** *Let* $\Delta := \Delta(a,b) \in [0,\pi]$. *For the base model* (62) *with* $\varphi \sim \mathrm{Unif}([0,2\pi))$,

$$E_{\mathrm{full}}(a,b) := \mathbb{E}_{\rho_0}\big[f_{ab}(\varphi)\big] = -1 + \frac{2\Delta}{\pi}. \tag{64}$$

Equivalently, writing

$$q_+(\Delta) := \Pr(f_{ab} = +1), \qquad q_-(\Delta) := \Pr(f_{ab} = -1), \tag{65}$$

one has

$$q_+(\Delta) = \frac{\Delta}{\pi}, \qquad q_-(\Delta) = 1 - \frac{\Delta}{\pi}. \tag{66}$$

Proof. Fix a, b and set $\Delta := \Delta(a,b) \in [0,\pi]$. By rotation invariance of $\varphi \sim \mathrm{Unif}([0,2\pi))$, we may shift φ by b and reduce to $b = 0$. The correlation depends only on the principal separation $\Delta(a,b)$, so we may represent $a - b$ by $\Delta \in [0,\pi]$.

Thus compute for

$$A(\Delta,\varphi) = \mathrm{sgn}(\cos(\varphi - \Delta)), \qquad B(0,\varphi) = -\,\mathrm{sgn}(\cos\varphi), \tag{67}$$

with $\varphi \sim \mathrm{Unif}([0,2\pi))$. Then

$$f_{\Delta,0}(\varphi) = -\,\mathrm{sgn}(\cos(\varphi - \Delta))\ \mathrm{sgn}(\cos\varphi). \tag{68}$$

Define the "positive-cosine" semicircle sets

$$P := \{\varphi \in [0,2\pi) : \cos\varphi \geq 0\}, \qquad P_\Delta := \{\varphi \in [0,2\pi) : \cos(\varphi - \Delta) \geq 0\}. \tag{69}$$

Then $f_{\Delta,0}(\varphi) = +1$ iff $\varphi \in P \triangle P_\Delta$, so

$$q_+(\Delta) = \Pr(f_{\Delta,0} = +1) = \frac{m(P \triangle P_\Delta)}{2\pi}, \tag{70}$$

where m denotes Lebesgue measure.

Represent

$$P = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \pmod{2\pi}, \qquad P_\Delta = \left[\Delta - \frac{\pi}{2}, \Delta + \frac{\pi}{2}\right] \pmod{2\pi}. \tag{71}$$

For $\Delta \in [0,\pi]$, the overlap has length $\pi - \Delta$, hence

$$m(P \cap P_\Delta) = \pi - \Delta. \tag{72}$$

Using $m(P \triangle P_\Delta) = m(P) + m(P_\Delta) - 2m(P \cap P_\Delta)$ with $m(P) = m(P_\Delta) = \pi$ gives

$$m(P \triangle P_\Delta) = 2\Delta. \tag{73}$$

Therefore $q_+(\Delta) = 2\Delta/(2\pi) = \Delta/\pi$ and $q_-(\Delta) = 1 - \Delta/\pi$.

Finally, since $f \in \{\pm 1\}$,

$$E_{\mathrm{full}}(a, b) = q_+(\Delta) - q_-(\Delta) = -1 + \frac{2\Delta}{\pi}. \tag{74}$$

Remark 4.3 (Unconditional CHSH is bounded by 2). The correlation (64) is generated by an unconditional Bell-local MI model, so it satisfies $S_{\mathrm{full}} \le 2$ on every quartet (Theorem 2.5). In particular, it cannot equal the singlet law (59) on a Tsirelson quartet, because (59) yields $S = 2\sqrt{2}$ on such a quartet.

4.3. Designing acceptance rates to convert the triangle into $-\cos$ *on accepted data*

We now show that the singlet law (59) can be reproduced from the base model (62) by a setting-dependent acceptance rule (rung-1 conditioning).

Enlarge the hidden space. Let

$$\Lambda := \Lambda_0 \times [0, 1] = [0, 2\pi) \times [0, 1], \qquad \lambda = (\varphi, u), \qquad \rho := \rho_0 \otimes \mathrm{Unif}([0, 1]). \tag{75}$$

The prior ρ is MI: it is independent of the settings (a, b).

Keep the same deterministic local responses

$$A(a, \varphi, u) := A(a, \varphi), \qquad B(b, \varphi, u) := B(b, \varphi), \tag{76}$$

and hence $f_{ab}(\varphi, u) := A(a, \varphi)B(b, \varphi) = f_{ab}(\varphi)$.

Acceptance depends only on the sign region of f_{ab}. Fix $\Delta \in (0, \pi)$. Accept with probability $\eta_+(\Delta)$ on the region $\{f_{ab} = +1\}$ and with probability $\eta_-(\Delta)$ on the region $\{f_{ab} = -1\}$, where $\eta_\pm(\Delta) \in (0, 1]$. Under such a rule, conditioning on acceptance reweights the two regions by factors proportional to $\eta_\pm(\Delta)$. A direct computation yields the accepted-sample correlator

$$E_{\mathrm{obs}}(a, b) = \frac{\eta_+(\Delta)\, q_+(\Delta) - \eta_-(\Delta)\, q_-(\Delta)}{\eta_+(\Delta)\, q_+(\Delta) + \eta_-(\Delta)\, q_-(\Delta)}. \tag{77}$$

Solve for the ratio. Let $E_{\mathrm{tgt}}(\Delta) := -\cos\Delta \in (-1, 1)$ for $\Delta \in (0, \pi)$. Introduce the ratio

$$r(\Delta) := \frac{\eta_+(\Delta)}{\eta_-(\Delta)} \in (0, \infty). \tag{78}$$

Solving $E_{\mathrm{obs}}(a, b) = E_{\mathrm{tgt}}(\Delta)$ for $r(\Delta)$ yields

$$r(\Delta) = \frac{q_-(\Delta)\,\big(1 + E_{\mathrm{tgt}}(\Delta)\big)}{q_+(\Delta)\,\big(1 - E_{\mathrm{tgt}}(\Delta)\big)} = \frac{\big(1 - \Delta/\pi\big)\,\big(1 - \cos\Delta\big)}{\big(\Delta/\pi\big)\,\big(1 + \cos\Delta\big)} = \frac{\pi - \Delta}{\Delta}\,\tan^2\!\Big(\frac{\Delta}{2}\Big). \tag{79}$$

Choose probabilities in $[0,1]$ with the required ratio. Define for $\Delta \in (0,\pi)$,

$$\eta_+(\Delta) := \min\{1, r(\Delta)\}, \qquad \eta_-(\Delta) := \min\{1, 1/r(\Delta)\}. \tag{80}$$

Then $\eta_\pm(\Delta) \in (0,1]$ and $\eta_+(\Delta)/\eta_-(\Delta) = r(\Delta)$.

At the endpoints $\Delta \in \{0,\pi\}$ the base model already satisfies $f_{ab} \equiv -1$ (for $\Delta = 0$) and $f_{ab} \equiv +1$ (for $\Delta = \pi$), hence $E_{\mathrm{full}} = E_{\mathrm{tgt}}$ automatically. We may therefore define

$$\eta_+(0) = \eta_-(0) = 1, \qquad \eta_+(\pi) = \eta_-(\pi) = 1. \tag{81}$$

4.4. A concrete acceptance map and the exact $-\cos$ theorem

We implement the acceptance probabilities $\eta_\pm(\Delta)$ by thresholding $u \sim \mathrm{Unif}([0,1])$.

Definition 4.4 (Setting-dependent acceptance map yielding $-\cos$)**.** For $(a,b) \in S_A \times S_B$ and $(\varphi,u) \in [0,2\pi)\times[0,1]$, let $\Delta = \Delta(a,b)$. Define the deterministic acceptance indicator

$$\gamma(a,b,\varphi,u) := \begin{cases} \mathbf{1}\{u \le \eta_+(\Delta)\}, & f_{ab}(\varphi) = +1, \\ \mathbf{1}\{u \le \eta_-(\Delta)\}, & f_{ab}(\varphi) = -1, \end{cases} \tag{82}$$

where $\eta_\pm$ are given by (80) and (81).

Remark 4.5 (Measurability and positivity). The map $(a,b,\varphi,u) \mapsto \gamma(a,b,\varphi,u)$ is measurable. Moreover, for every (a,b) the acceptance rate $Z(a,b) = \int \gamma\, d\rho$ is strictly positive: for $\Delta \in (0,\pi)$ both $q_\pm(\Delta) > 0$ and both $\eta_\pm(\Delta) > 0$, and for $\Delta \in \{0,\pi\}$ one has $Z(a,b) = 1$. Thus Definition 2.9 applies.

Proposition 4.6 (The acceptance map is not locally factorizable)**.** *In the acceptance construction of Definition 4.4, there do not exist measurable functions*

$$\gamma_A : S_A \times \Lambda \to [0,1], \qquad \gamma_B : S_B \times \Lambda \to [0,1], \tag{83}$$

such that

$$\gamma(a,b,\lambda) = \gamma_A(a,\lambda)\,\gamma_B(b,\lambda) \quad \textit{for all } (a,b,\lambda) \in S_A \times S_B \times \Lambda, \tag{84}$$

or even ρ-almost surely.

Proof. Assume for contradiction that such γ_A, γ_B exist (even ρ-a.s.). Choose parameters $0 < \delta < \varepsilon < \pi/16$ and define four settings

$$a_0 = \frac{\pi}{2} - \varepsilon, \qquad a_1 = \frac{\pi}{2} + \varepsilon, \qquad b_0 = \frac{\pi}{2} + \varepsilon + \delta, \qquad b_1 = \frac{\pi}{2} - \varepsilon + \delta. \tag{85}$$

Consider $\Lambda = [0,2\pi) \times [0,1]$ with $\lambda = (\varphi,u)$. Let $I := [0, (\varepsilon-\delta)/2] \subset [0,2\pi)$, which has positive Lebesgue measure. For every $\varphi \in I$, the signs satisfy

$$\mathrm{sgn}(\cos(\varphi - a_0)) = +1, \quad \mathrm{sgn}(\cos(\varphi - a_1)) = -1, \quad \mathrm{sgn}(\cos(\varphi - b_0)) = -1, \quad \mathrm{sgn}(\cos(\varphi - b_1)) = +1 \tag{86}$$

hence under (62) one has

$$f_{a_0b_0}(\varphi) = +1, \quad f_{a_0b_1}(\varphi) = -1, \quad f_{a_1b_0}(\varphi) = -1, \quad f_{a_1b_1}(\varphi) = +1. \tag{87}$$

Moreover,

$$\Delta(a_0, b_0) = 2\varepsilon + \delta \in (0, \pi/2), \qquad \Delta(a_1, b_1) = 2\varepsilon - \delta \in (0, \pi/2), \tag{88}$$

so $r(\Delta) < 1$ and therefore $\eta_-(\Delta) = 1$ and $\eta_+(\Delta) = r(\Delta) \in (0,1)$ for these two pairs. Let

$$u_\star := \max\{\eta_+(\Delta(a_0, b_0)), \eta_+(\Delta(a_1, b_1))\} < 1, \tag{89}$$

and choose $u \in (u_\star, 1]$ (a set of positive measure in $[0,1]$).

For every $\lambda = (\varphi, u) \in I \times (u_\star, 1]$ the acceptance rule (82) then gives

$$\gamma(a_0, b_0, \lambda) = 0, \quad \gamma(a_0, b_1, \lambda) = 1, \quad \gamma(a_1, b_0, \lambda) = 1, \quad \gamma(a_1, b_1, \lambda) = 0, \tag{90}$$

i.e. the 2×2 table is

$$\big(\gamma(a_i, b_j, \lambda)\big)_{i,j\in\{0,1\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{91}$$

If γ factorized as $\gamma_A(a, \lambda)\gamma_B(b, \lambda)$, then the table would have rank 1 and must satisfy

$$\gamma(a_0, b_0, \lambda)\, \gamma(a_1, b_1, \lambda) = \gamma(a_0, b_1, \lambda)\, \gamma(a_1, b_0, \lambda), \tag{92}$$

but here the left-hand side is 0 and the right-hand side is 1, a contradiction. Since the set $I \times (u_\star, 1]$ has positive ρ-measure, this contradicts even ρ-a.s. factorization.

Theorem 4.7 (Exact singlet correlator on accepted data from a Bell-local deterministic base model)**.** *Let ρ be the MI prior on $\Lambda = [0, 2\pi) \times [0, 1]$ defined above, let A, B be the deterministic local outputs* (62)*, and let γ be the acceptance map* (82)*. Let ν_{ab} be the accepted laws induced by γ via* (14)*. Then the accepted-sample correlator satisfies*

$$E_{\text{obs}}(a, b) = \int_\Lambda A(a, \lambda)B(b, \lambda)\, d\nu_{ab}(\lambda) = -\cos(a - b) \qquad \textit{for all } a, b \in \mathbb{R}/2\pi\mathbb{Z}. \tag{93}$$

Proof. Fix (a, b) and abbreviate $\Delta := \Delta(a, b) \in [0, \pi]$. Write $f(\varphi) := f_{ab}(\varphi) \in \{\pm 1\}$ and view γ as $\gamma(\varphi, u)$.

Step 1: compute the acceptance rate. By definition (82), conditional on φ,

$$\Pr(\gamma = 1 \mid \varphi) = \begin{cases} \eta_+(\Delta), & f(\varphi) = +1, \\ \eta_-(\Delta), & f(\varphi) = -1, \end{cases} \tag{94}$$

because $u \sim \text{Unif}([0, 1])$ is independent of φ and $\Pr(u \le t) = t$ for $t \in [0, 1]$. Therefore, with $q_\pm(\Delta)$ from Proposition 4.2,

$$Z(a, b) = \Pr(\gamma = 1) = \eta_+(\Delta)\, q_+(\Delta) + \eta_-(\Delta)\, q_-(\Delta). \tag{95}$$

Step 2: compute the accepted-sample correlator as a conditional expectation. Since A, B are deterministic $\{\pm 1\}$ outputs, $AB = f(\varphi)$ and $\gamma \in \{0, 1\}$, and

$$E_{\text{obs}}(a, b) = \mathbb{E}[\, f(\varphi) \mid \gamma = 1\,] = \frac{\mathbb{E}[f(\varphi)\gamma(\varphi, u)]}{\mathbb{E}[\gamma(\varphi, u)]}. \tag{96}$$

Condition on φ:

$$\mathbb{E}[f\gamma] = \mathbb{E}\big[\mathbb{E}[f\gamma \mid \varphi]\big] = \mathbb{E}\big[f(\varphi)\,\mathbb{E}[\gamma \mid \varphi]\big] = \eta_+(\Delta)\, q_+(\Delta) - \eta_-(\Delta)\, q_-(\Delta). \tag{97}$$

Dividing by $Z(a, b)$ from (95) gives (77).

Step 3: enforce the target by the chosen ratio. For $\Delta \in (0, \pi)$ define $r(\Delta) := \eta_+(\Delta)/\eta_-(\Delta)$. Then

$$E_{\text{obs}}(a, b) = \frac{r(\Delta)\, q_+(\Delta) - q_-(\Delta)}{r(\Delta)\, q_+(\Delta) + q_-(\Delta)}. \tag{98}$$

Solving $E_{\text{obs}} = E_{\text{tgt}}(\Delta) = -\cos\Delta$ yields

$$r(\Delta) = \frac{q_-(\Delta)\,\big(1 + E_{\text{tgt}}(\Delta)\big)}{q_+(\Delta)\,\big(1 - E_{\text{tgt}}(\Delta)\big)}, \tag{99}$$

which is exactly (79). The $\eta_\pm$ in (80) satisfy $\eta_+/\eta_- = r$, hence $E_{\text{obs}}(a, b) = -\cos\Delta = -\cos(a - b)$.

For $\Delta \in \{0, \pi\}$, the base model has $f \equiv -1$ (at $\Delta = 0$) and $f \equiv +1$ (at $\Delta = \pi$), so conditioning on any non-null acceptance rule yields the same constant correlation, which equals $-\cos\Delta$.

4.5. Acceptance-rate diagnostics: closed form for $Z(\Delta)$ and sample values

Recall the acceptance rate

$$Z(a, b) = \int_\Lambda \gamma(a, b, \lambda)\, d\rho(\lambda) = \Pr_\rho(\gamma = 1 \mid a, b). \tag{100}$$

In the present construction $Z(a, b)$ depends only on the principal separation $\Delta = \Delta(a, b)$.

Lemma 4.8 (Monotonicity and branch location for $r(\Delta)$)**.** *Let*

$$r(\Delta) := \frac{\pi - \Delta}{\Delta}\, \tan^2\!\Big(\frac{\Delta}{2}\Big), \qquad \Delta \in (0, \pi). \tag{101}$$

Then:

(i) $r(\pi - \Delta) = 1/r(\Delta)$ *for all* $\Delta \in (0, \pi)$.

(ii) r *is strictly increasing on* $(0, \pi/2)$*, with* $\lim_{\Delta \downarrow 0} r(\Delta) = 0$ *and* $r(\pi/2) = 1$. *Consequently,*

$$r(\Delta) < 1 \textit{ for } \Delta \in (0, \pi/2), \qquad r(\Delta) > 1 \textit{ for } \Delta \in (\pi/2, \pi). \tag{102}$$

Proof. (i) Using $\tan\left(\frac{\pi-\Delta}{2}\right) = \cot(\Delta/2)$,

$$r(\pi - \Delta) = \frac{\Delta}{\pi - \Delta}\cot^2\left(\frac{\Delta}{2}\right) = \frac{1}{r(\Delta)}. \tag{103}$$

(ii) Differentiate $\log r(\Delta) = \log(\pi - \Delta) - \log \Delta + 2 \log \tan(\Delta/2)$ on $(0, \pi)$:

$$\frac{d}{d\Delta}\log r(\Delta) = -\frac{1}{\pi-\Delta} - \frac{1}{\Delta} + 2\cdot\frac{d}{d\Delta}\log\tan(\Delta/2) = -\frac{1}{\pi-\Delta} - \frac{1}{\Delta} + \frac{2}{\sin\Delta}. \tag{104}$$

For $\Delta \in (0, \pi/2)$, we have $\sin\Delta < \Delta$, hence $\frac{2}{\sin\Delta} > \frac{2}{\Delta}$, and also $\pi - \Delta > \Delta$, hence $\frac{1}{\pi-\Delta} < \frac{1}{\Delta}$. Therefore

$$\frac{2}{\sin\Delta} - \frac{1}{\Delta} - \frac{1}{\pi-\Delta} > \frac{2}{\Delta} - \frac{1}{\Delta} - \frac{1}{\Delta} = 0, \tag{105}$$

so $(\log r)'(\Delta) > 0$ and thus r is strictly increasing on $(0, \pi/2)$.

Next,

$$\lim_{\Delta\downarrow 0} r(\Delta) = \lim_{\Delta\downarrow 0}\frac{\pi-\Delta}{\Delta}\left(\frac{\Delta}{2} + o(\Delta)\right)^2 = \lim_{\Delta\downarrow 0}\frac{\pi-\Delta}{\Delta}\cdot\frac{\Delta^2}{4} = 0, \tag{106}$$

and

$$r(\pi/2) = \frac{\pi - \pi/2}{\pi/2}\tan^2(\pi/4) = 1. \tag{107}$$

Strict increase plus the endpoint values implies $r(\Delta) < 1$ on $(0, \pi/2)$; then (i) implies $r(\Delta) > 1$ on $(\pi/2, \pi)$.

Proposition 4.9 (Closed form of the acceptance rate $Z(\Delta)$)**.** *In the acceptance model of Definition 4.4, the acceptance rate depends only on $\Delta \in [0, \pi]$ and is given by*

$$Z(\Delta) = \begin{cases} \dfrac{\pi-\Delta}{\pi}\sec^2\left(\dfrac{\Delta}{2}\right), & 0 \le \Delta \le \frac{\pi}{2}, \\ \dfrac{\Delta}{\pi}\csc^2\left(\dfrac{\Delta}{2}\right), & \frac{\pi}{2} \le \Delta \le \pi. \end{cases} \tag{108}$$

Equivalently,

$$Z(\Delta) = \min\left\{\frac{\pi-\Delta}{\pi\cos^2(\Delta/2)}, \frac{\Delta}{\pi\sin^2(\Delta/2)}\right\}. \tag{109}$$

Moreover, $Z(0) = Z(\pi/2) = Z(\pi) = 1$, the function is symmetric $Z(\Delta) = Z(\pi - \Delta)$, and

$$\inf_{\Delta\in[0,\pi]} Z(\Delta) \approx 0.8786 \quad \textit{(attained near } \Delta \approx 0.8105 \textit{ rad } (\approx 46.4^\circ) \textit{ and symmetrically near } \pi-\Delta). \tag{110}$$

Proof. Fix $\Delta \in [0, \pi]$ and recall from Proposition 4.2 that $q_+(\Delta) = \Delta/\pi$ and $q_-(\Delta) = (\pi - \Delta)/\pi$. By construction,

$$Z(\Delta) = \eta_+(\Delta)\, q_+(\Delta) + \eta_-(\Delta)\, q_-(\Delta), \tag{111}$$

where $r(\Delta) = \eta_+(\Delta)/\eta_-(\Delta)$ is given by (79).

For $\Delta \in (0, \pi)$, Lemma 4.8 shows that $r(\Delta) \le 1$ exactly on $(0, \pi/2]$ and $r(\Delta) \ge 1$ exactly on $[\pi/2, \pi)$; we treat the two cases accordingly:

Case 1: $r(\Delta) \le 1$. Then $\eta_+(\Delta) = r(\Delta)$ and $\eta_-(\Delta) = 1$, so

$$\begin{aligned} Z(\Delta) &= r(\Delta)\frac{\Delta}{\pi} + \frac{\pi - \Delta}{\pi} \\ &= \frac{\pi - \Delta}{\pi}\tan^2\Big(\frac{\Delta}{2}\Big) + \frac{\pi - \Delta}{\pi} \\ &= \frac{\pi - \Delta}{\pi}\Big(1 + \tan^2\Big(\frac{\Delta}{2}\Big)\Big) \\ &= \frac{\pi - \Delta}{\pi}\sec^2\Big(\frac{\Delta}{2}\Big). \end{aligned} \tag{112}$$

Case 2: $r(\Delta) \ge 1$. Then $\eta_+(\Delta) = 1$ and $\eta_-(\Delta) = 1/r(\Delta)$, so

$$\begin{aligned} Z(\Delta) &= \frac{\Delta}{\pi} + \frac{1}{r(\Delta)}\frac{\pi - \Delta}{\pi} \\ &= \frac{\Delta}{\pi} + \frac{\Delta}{\pi}\cot^2\Big(\frac{\Delta}{2}\Big) \\ &= \frac{\Delta}{\pi}\Big(1 + \cot^2\Big(\frac{\Delta}{2}\Big)\Big) \\ &= \frac{\Delta}{\pi}\csc^2\Big(\frac{\Delta}{2}\Big). \end{aligned} \tag{113}$$

Lemma 4.8 gives $r(\pi/2) = 1$ and $r(\pi - \Delta) = 1/r(\Delta)$, so the two displayed branches meet at $\Delta = \pi/2$. Moreover, since $q_+(\pi - \Delta) = q_-(\Delta)$ and $q_-(\pi - \Delta) = q_+(\Delta)$, the definition of $Z(\Delta)$ implies the symmetry $Z(\Delta) = Z(\pi - \Delta)$.

The endpoint values $Z(0) = Z(\pi) = 1$ follow from the endpoint convention (81) (equivalently, from $q_+(0) = 0$ and $q_-(\pi) = 0$). The stated numerical minimum follows from elementary calculus applied to the first branch on $[0, \pi/2]$: the critical point satisfies $(\pi - \Delta)\tan(\Delta/2) = 1$, yielding $\Delta \approx 0.8105$ and $Z(\Delta) \approx 0.8786$.

Corollary 4.10 (Acceptance-rate signature on the standard Tsirelson quartet)**.** *For the standard Tsirelson quartet*

$$a_0 = 0, \quad a_1 = \frac{\pi}{2}, \quad b_0 = \frac{\pi}{4}, \quad b_1 = -\frac{\pi}{4}, \tag{114}$$

the four setting separations are $\Delta_{00} = \Delta_{01} = \Delta_{10} = \pi/4$ *and* $\Delta_{11} = 3\pi/4$*. By symmetry* $Z(3\pi/4) = Z(\pi/4)$*, so the acceptance rates are equal across the quartet:*

$$Z_{00} = Z_{01} = Z_{10} = Z_{11} = Z\Big(\frac{\pi}{4}\Big) = 3 - \frac{3}{2}\sqrt{2} \approx 0.878679656. \tag{115}$$

Representative numerical values of the acceptance rate are listed in Table 1.

Remark 4.11 (Plot description)*.* The curve $\Delta \mapsto Z(\Delta)$ on $[0, \pi]$ is symmetric about $\pi/2$, equals 1 at $\Delta = 0, \pi/2, \pi$, and exhibits a shallow dip with minimum $\inf_\Delta Z(\Delta) \approx 0.8786$ near $\Delta \approx 46.4^\circ$ and (by symmetry) near 133.6°. Thus the rejection fraction $1 - Z(\Delta)$ never exceeds approximately 12.2% in this construction.

Table 1. Acceptance rate $Z(\Delta)$ at common angular separations (degrees shown for convenience; Δ is in radians in the theory).

Δ (deg)	Δ (rad)	Exact $Z(\Delta)$	$Z(\Delta)$
$0°$	0	1	1.000000000
$22.5°$	$\pi/8$	$\frac{7}{8}\sec^2(\pi/16) = \frac{7}{2(2+\sqrt{2+\sqrt{2}})}$	0.909620364
$45°$	$\pi/4$	$\frac{3}{4}\sec^2(\pi/8) = 3 - \frac{3\sqrt{2}}{2}$	0.878679656
$67.5°$	$3\pi/8$	$\frac{5}{8}\sec^2(3\pi/16) = \frac{5}{2(2+\sqrt{2-\sqrt{2}})}$	0.904039183
$90°$	$\pi/2$	1	1.000000000

Lemma 4.12 (Accepted marginals are unbiased)**.** *In the setting-dependent acceptance model of Theorem 4.7, for every* (a, b) *one has*

$$\mathbb{E}_{\nu_{ab}}[A(a,\lambda)] = 0, \qquad \mathbb{E}_{\nu_{ab}}[B(b,\lambda)] = 0. \tag{116}$$

Equivalently, under ν_{ab}*,*

$$\Pr(A = +1) = \Pr(A = -1) = \Pr(B = +1) = \Pr(B = -1) = \frac{1}{2}. \tag{117}$$

Proof. Fix (a, b) and write $\Delta = \Delta(a, b)$. Let $T : [0, 2\pi) \times [0, 1] \to [0, 2\pi) \times [0, 1]$ be the measure-preserving involution

$$T(\varphi, u) := (\varphi + \pi \bmod 2\pi,\ u). \tag{118}$$

Step 1: sign-flip identities hold a.e. Because sgn is defined with $\operatorname{sgn}(0) = +1$, the identities $A(a, \varphi + \pi) = -A(a, \varphi)$ and $B(b, \varphi + \pi) = -B(b, \varphi)$ can fail only when the relevant cosine equals 0. Define the null sets

$$N_A := \{\varphi \in [0, 2\pi) : \cos(\varphi - a) = 0\}, \qquad N_B := \{\varphi \in [0, 2\pi) : \cos(\varphi - b) = 0\}. \tag{119}$$

Each is finite, hence Lebesgue-null. Thus for Lebesgue-a.e. φ one has

$$A(a, \varphi + \pi) = -A(a, \varphi), \qquad B(b, \varphi + \pi) = -B(b, \varphi), \tag{120}$$

and therefore for Lebesgue-a.e. φ,

$$f_{ab}(\varphi + \pi) = A(a, \varphi + \pi)B(b, \varphi + \pi) = A(a, \varphi)B(b, \varphi) = f_{ab}(\varphi). \tag{121}$$

Step 2: γ is invariant under T a.e. Since γ in (82) depends on (φ, u) only through $f_{ab}(\varphi) \in \{\pm 1\}$ and the threshold comparison with u, the identity $f_{ab}(\varphi + \pi) = f_{ab}(\varphi)$ implies

$$\gamma(a, b, T(\varphi, u)) = \gamma(a, b, \varphi, u) \qquad \text{for } \rho\text{-a.e. } (\varphi, u). \tag{122}$$

Step 3: conclude unbiasedness. By the accepted-law formula (14),

$$\mathbb{E}_{\nu_{ab}}[A] = \frac{1}{Z(a,b)} \int A(a,\varphi)\,\gamma(a,b,\varphi,u)\,d\rho(\varphi,u). \tag{123}$$

Apply the change of variables $(\varphi, u) \mapsto T(\varphi, u)$ (which preserves ρ), using the a.e. identities above:

$$\begin{aligned}\mathbb{E}_{\nu_{ab}}[A] &= \frac{1}{Z(a,b)} \int A(a,T(\varphi,u))\,\gamma(a,b,T(\varphi,u))\,d\rho(\varphi,u)\\ &= \frac{1}{Z(a,b)} \int \big(-A(a,\varphi)\big)\,\gamma(a,b,\varphi,u)\,d\rho(\varphi,u)\\ &= -\mathbb{E}_{\nu_{ab}}[A].\end{aligned} \tag{124}$$

Hence $\mathbb{E}_{\nu_{ab}}[A] = 0$. The same argument gives $\mathbb{E}_{\nu_{ab}}[B] = 0$.

Since $A, B \in \{\pm 1\}$, expectation 0 implies each takes ± 1 with probability $1/2$.

Corollary 4.13 (Full accepted-sample singlet joint law)**.** *In the setting-dependent acceptance model of Theorem 4.7, for every a, b and every $\alpha, \beta \in \{\pm 1\}$,*

$$\Pr_{\nu_{ab}}\big(A = \alpha,\ B = \beta\big) = \frac{1}{4}\Big(1 - \alpha\beta\cos(a-b)\Big). \tag{125}$$

In particular, the accepted-sample $\{\pm 1\}$ outcomes are no-signalling at the level of marginal distributions: $P_{\nu_{ab}}(A = \pm 1) = P_{\nu_{ab'}}(A = \pm 1) = 1/2$ and similarly for Bob.

Proof. Fix (a, b) and abbreviate $E := \mathbb{E}_{\nu_{ab}}[AB]$. By Lemma 4.12, $\mathbb{E}_{\nu_{ab}}[A] = \mathbb{E}_{\nu_{ab}}[B] = 0$. For $\alpha, \beta \in \{\pm 1\}$, the four probabilities satisfy the linear system

$$\Pr(A = \alpha, B = \beta) = \frac{1}{4}\Big(1 + \alpha\,\mathbb{E}[A] + \beta\,\mathbb{E}[B] + \alpha\beta\,\mathbb{E}[AB]\Big), \tag{126}$$

hence (using $\mathbb{E}[A] = \mathbb{E}[B] = 0$) one obtains

$$\Pr(A = \alpha, B = \beta) = \frac{1}{4}\Big(1 + \alpha\beta\,E\Big). \tag{127}$$

Now apply Theorem 4.7, which gives $E = -\cos(a - b)$.

Remark 4.14 (Why this does not contradict Bell–CHSH). The construction preserves Bell locality of $A(a, \lambda)$ and $B(b, \lambda)$ and measurement independence at emission (the prior ρ is fixed and does not depend on settings). What changes is the *ensemble* on which correlators are evaluated: for each setting pair (a, b) the correlator (and joint law) is evaluated under the accepted law ν_{ab}, and the family $\{\nu_{ab}\}$ is setting dependent. Hence one is in rung-1 semantics, not the unconditional rung-0 Bell class.

Remark 4.15 (Selection, loopholes, and observability of rejection). Corollary 4.13 shows that the *postselected* $\{\pm 1\}$ outcomes can be arranged to be no-signalling. However, the *acceptance event* itself is setting dependent in general; if acceptance/rejection information is locally accessible, it can carry setting dependence. Moreover, in the explicit construction here the acceptance rule $\gamma(a, b, \lambda)$ is not locally factorizable as $\gamma_A(a, \lambda)\gamma_B(b, \lambda)$ (Proposition 4.6), so it cannot be interpreted as independent local detection inefficiencies; it is intrinsically a joint/coincidence-style filter. This is precisely the structural reason postselection is treated as a loophole rather than an unconditional Bell-local explanation [3–7].

Remark 4.16 (Scope of the construction). The selection model is presented as a semantic/measure-theoretic realization of how setting-dependent conditioning can reproduce Tsirelson-scale correlators from Bell-local deterministic base responses. It is not a claim about loophole-free Bell experiments, where acceptance mechanisms are constrained by spacelike separation, device characterization, and explicit event-ready or high-efficiency detection protocols.

4.6. Rung-0 no-go: no unconditional Bell-local MI model can reproduce a Tsirelson $-\cos$ *table on a Tsirelson quartet*

Proposition 4.17 (Unconditional rung-0 no-go for Tsirelson $-\cos$)**.** *Fix a CHSH quartet* (a_0, a_1, b_0, b_1) *such that the target table*

$$E_{ij}^{\mathrm{tgt}} := -\cos(a_i - b_j) \tag{128}$$

yields

$$S^{\mathrm{tgt}} := \left|E_{00}^{\mathrm{tgt}} + E_{01}^{\mathrm{tgt}} + E_{10}^{\mathrm{tgt}} - E_{11}^{\mathrm{tgt}}\right| = 2\sqrt{2}. \tag{129}$$

Then there do not exist:

- *a probability space* $(\Lambda, \mathcal{F}, \rho)$ *with* $\rho \in \mathcal{P}(\Lambda)$*,*
- *bounded measurable functions* $A_i, B_j : \Lambda \to [-1, 1]$*,*

such that $E_{ij}^{\mathrm{tgt}} = \int A_i(\lambda)B_j(\lambda)\, d\rho(\lambda)$ *for all* $i, j \in \{0, 1\}$*.*

Proof. Assume such (Λ, ρ) and A_i, B_j exist. Define

$$C(\lambda) := A_0(\lambda)B_0(\lambda) + A_0(\lambda)B_1(\lambda) + A_1(\lambda)B_0(\lambda) - A_1(\lambda)B_1(\lambda). \tag{130}$$

Pointwise, $|C(\lambda)| \le 2$ by Lemma 2.4. Therefore

$$2\sqrt{2} = \left|\int_\Lambda C(\lambda)\, d\rho(\lambda)\right| \le \int_\Lambda |C(\lambda)|\, d\rho(\lambda) \le \int_\Lambda 2\, d\rho(\lambda) = 2, \tag{131}$$

a contradiction.

4.7. Optional: a single-sample-space representation becomes possible with a signed *(quasi-probability) measure*

Because PV discussions sometimes appeal to "non-Boolean" or "non-classical" semantics, it is useful to record one precise fact: although a *probability* measure cannot reproduce Tsirelson correlations in a deterministic rung-0 model, a *signed* measure can.

Let $\Omega := \{\pm 1\}^4$ with elements $\omega = (a_0, a_1, b_0, b_1)$ interpreted as deterministic outcome assignments on one fixed CHSH quartet. Define coordinate functions

$$A_0(\omega) := a_0, \quad A_1(\omega) := a_1, \quad B_0(\omega) := b_0, \quad B_1(\omega) := b_1. \tag{132}$$

Define the pointwise CHSH sum

$$T(\omega) := a_0 b_0 + a_0 b_1 + a_1 b_0 - a_1 b_1 \in \{\pm 2\}. \tag{133}$$

Let

$$t := \frac{\sqrt{2}}{2}. \tag{134}$$

Proposition 4.18 (Explicit signed-measure representation of the Tsirelson correlation table on one quartet)**.** *Define a signed measure $\widetilde{\rho}$ on Ω by*

$$\widetilde{\rho}(\{\omega\}) := \frac{1 - t\,T(\omega)}{16}. \tag{135}$$

Then:

(i) (Normalization) $\sum_{\omega \in \Omega} \widetilde{\rho}(\{\omega\}) = 1$.

(ii) (Negativity) $\widetilde{\rho}$ takes negative values; indeed $\widetilde{\rho}(\omega) = (1 - \sqrt{2})/16 < 0$ when $T(\omega) = +2$.

(iii) (Tsirelson moments) The pairwise moments reproduce the standard Tsirelson table

$$\mathbb{E}_{\widetilde{\rho}}[A_0 B_0] = \mathbb{E}_{\widetilde{\rho}}[A_0 B_1] = \mathbb{E}_{\widetilde{\rho}}[A_1 B_0] = -t, \qquad \mathbb{E}_{\widetilde{\rho}}[A_1 B_1] = +t, \tag{136}$$

and hence the CHSH value equals $2\sqrt{2}$.

Proof. Write U for the uniform probability measure on Ω, i.e. $U(\omega) = 1/16$. From (135),

$$\widetilde{\rho}(\omega) = U(\omega)\,(1 - tT(\omega)). \tag{137}$$

Hence for any function $F : \Omega \to \mathbb{R}$,

$$\mathbb{E}_{\widetilde{\rho}}[F] = \mathbb{E}_U[F] - t\,\mathbb{E}_U[FT]. \tag{138}$$

(i) Normalization. Take $F \equiv 1$ in (138):

$$\sum_{\omega} \widetilde{\rho}(\omega) = \mathbb{E}_{\widetilde{\rho}}[1] = \mathbb{E}_U[1] - t\,\mathbb{E}_U[T] = 1 - t\,\mathbb{E}_U[T]. \tag{139}$$

Under U the variables a_0, a_1, b_0, b_1 are independent symmetric $\{\pm 1\}$, so each product $a_0 b_0$, $a_0 b_1$, $a_1 b_0$, $a_1 b_1$ has mean 0. Therefore $\mathbb{E}_U[T] = 0$ and the sum is 1.

(ii) Negativity. Since $T(\omega) \in \{\pm 2\}$,

$$\widetilde{\rho}(\omega) = \frac{1-2t}{16} = \frac{1-\sqrt{2}}{16} < 0 \quad \text{if } T(\omega) = +2, \qquad \widetilde{\rho}(\omega) = \frac{1+2t}{16} = \frac{1+\sqrt{2}}{16} > 0 \tag{140}$$

if $T(\omega) = -2$.

(iii) Pairwise moments. Let $F(\omega) = A_i(\omega) B_j(\omega) = a_i b_j$. Then $\mathbb{E}_U[F] = 0$ by symmetry, so (138) gives

$$\mathbb{E}_{\widetilde{\rho}}[a_i b_j] = -t\,\mathbb{E}_U[a_i b_j\, T]. \tag{141}$$

One checks by direct expansion and independence under U that $\mathbb{E}_U[a_0 b_0\, T] = \mathbb{E}_U[a_0 b_1\, T] = \mathbb{E}_U[a_1 b_0\, T] = 1$ and $\mathbb{E}_U[a_1 b_1\, T] = -1$. Hence

$$\mathbb{E}_{\widetilde{\rho}}[A_0 B_0] = \mathbb{E}_{\widetilde{\rho}}[A_0 B_1] = \mathbb{E}_{\widetilde{\rho}}[A_1 B_0] = -t, \qquad \mathbb{E}_{\widetilde{\rho}}[A_1 B_1] = +t, \tag{142}$$

which yields CHSH $= 2\sqrt{2}$.

Remark 4.19 (Why this does not contradict Bell). Bell–CHSH assumes ρ is a *probability measure* (positive). Proposition 4.18 drops positivity and therefore leaves the Kolmogorov/Boolean hypothesis class.

5. A PV-Indexed Microcausal (Noncommutative) Realization of the Singlet Law

5.1. Microcausal operator-algebraic framework and Tsirelson bound

The rung-0 (Bell/CHSH classical) hypothesis class is a commutative (Kolmogorov) probability theory in which all counterfactuals live on one probability space. A distinct and logically independent locality notion is *microcausality* in operator-algebraic models: Alice and Bob observable algebras commute, but the algebra need not be commutative within each wing. In this noncommutative class, the sharp universal CHSH upper bound is Tsirelson's $2\sqrt{2}$ [8,9].

Definition 5.1 (C*-probability space)**.** A C^**-probability space* is a pair $(\mathcal{A}, \omega)$ where $\mathcal{A}$ is a unital C*-algebra and $\omega : \mathcal{A} \to \mathbb{C}$ is a state (linear, positive, and normalized: $\omega(1_{\mathcal{A}}) = 1$).

Definition 5.2 (Microcausal bipartite structure)**.** Let $(\mathcal{A}, \omega)$ be a C*-probability space. A *microcausal bipartite structure* consists of commuting unital C*-subalgebras $\mathcal{A}_A, \mathcal{A}_B \subset \mathcal{A}$ such that

$$[X, Y] = 0 \qquad (\forall X \in \mathcal{A}_A,\ \forall Y \in \mathcal{A}_B), \tag{143}$$

where $[X, Y] := XY - YX$.

Definition 5.3 (CHSH in a microcausal C* model)**.** Fix self-adjoint unitaries

$$A_0, A_1 \in \mathcal{A}_A, \qquad B_0, B_1 \in \mathcal{A}_B, \qquad A_i^* = A_i,\ B_j^* = B_j,\ A_i^2 = B_j^2 = 1_{\mathcal{A}}. \tag{144}$$

Define correlators $E_{ij} := \omega(A_i B_j)$ and the CHSH value

$$S_\omega := \big|E_{00} + E_{01} + E_{10} - E_{11}\big| = \big|\omega(A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1)\big|. \tag{145}$$

Theorem 5.4 (Tsirelson bound under microcausality)**.** *In any microcausal C* model in the sense of Definitions 5.1–5.3,*

$$S_\omega \le 2\sqrt{2}. \tag{146}$$

Moreover, if $[A_0, A_1] = 0$ *or* $[B_0, B_1] = 0$ *then* $S_\omega \le 2$.

Proof. Define the Bell operator

$$\mathcal{B} := A_0(B_0 + B_1) + A_1(B_0 - B_1) \in \mathcal{A}. \tag{147}$$

Then $S_\omega = |\omega(\mathcal{B})| \le \|\mathcal{B}\|$ since $|\omega(X)| \le \|X\|$ for any state ω.

Step 1: compute $\mathcal{B}^2$. Using microcausality $[A_i, B_j] = 0$, expand:

$$\mathcal{B}^2 = A_0^2(B_0 + B_1)^2 + A_1^2(B_0 - B_1)^2 + A_0A_1(B_0 + B_1)(B_0 - B_1) + A_1A_0(B_0 - B_1)(B_0 + B_1). \tag{148}$$

Since $A_0^2 = A_1^2 = 1_{\mathcal{A}}$,

$$\mathcal{B}^2 = (B_0+B_1)^2 + (B_0-B_1)^2 + A_0A_1(B_0+B_1)(B_0-B_1) + A_1A_0(B_0-B_1)(B_0+B_1). \tag{149}$$

Now

$$(B_0 + B_1)^2 + (B_0 - B_1)^2 = 2(B_0^2 + B_1^2) = 4\,1_{\mathcal{A}}, \tag{150}$$

and

$$(B_0 + B_1)(B_0 - B_1) = -[B_0, B_1], \qquad (B_0 - B_1)(B_0 + B_1) = [B_0, B_1]. \tag{151}$$

Therefore the cross terms equal

$$A_0A_1(-[B_0, B_1]) + A_1A_0([B_0, B_1]) = -(A_0A_1 - A_1A_0)[B_0, B_1] = -[A_0, A_1]\,[B_0, B_1]. \tag{152}$$

Hence

$$\mathcal{B}^2 = 4\,1_{\mathcal{A}} - [A_0, A_1]\,[B_0, B_1]. \tag{153}$$

Step 2: bound the norm. Because $\mathcal{B}$ is self-adjoint, $\|\mathcal{B}\|^2 = \|\mathcal{B}^2\|$. From (153),

$$\|\mathcal{B}^2\| \le 4 + \|[A_0, A_1]\,[B_0, B_1]\| \le 4 + \|[A_0, A_1]\|\,\|[B_0, B_1]\|. \tag{154}$$

Since $\|A_i\| = \|B_j\| = 1$ (self-adjoint unitaries),

$$\|[A_0, A_1]\| \le \|A_0 A_1\| + \|A_1 A_0\| \le 2, \qquad \|[B_0, B_1]\| \le 2. \tag{155}$$

Thus $\|\mathcal{B}^2\| \le 8$, so $\|\mathcal{B}\| \le 2\sqrt{2}$ and therefore $S_\omega \le 2\sqrt{2}$.

Step 3: commutative subcase. If $[A_0, A_1] = 0$ or $[B_0, B_1] = 0$, then (153) gives $\mathcal{B}^2 = 4\,1_{\mathcal{A}}$. Hence $\|\mathcal{B}\| = 2$ and $S_\omega \le 2$.

Remark 5.5 (Relation to Bell locality). Theorem 5.4 does *not* contradict Theorem 2.5. The latter is a theorem in a *commutative* (Kolmogorov) probability model where all observables are jointly representable as random variables on one probability space. Microcausal operator-algebraic models are generally noncommutative and therefore lie outside that hypothesis class.

5.2. PV semicircle parameter and a canonical $SU(2)$ unitary

Fix $c > 0$. Let the phenomenological velocity satisfy $v \in (-c, c)$ and define

$$\kappa(v) := \sqrt{1 - \frac{v^2}{c^2}} \in (0, 1]. \tag{156}$$

To avoid collision with the hidden-variable angle φ used in Section 4, we denote the PV semicircle angle by $\vartheta(v) \in (0, \pi)$ defined by

$$\cos\vartheta(v) = \frac{v}{c}, \qquad \sin\vartheta(v) = \kappa(v). \tag{157}$$

Pauli matrices. We use

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{158}$$

Definition 5.6 (PV-indexed unitary)**.** Define

$$U(v) := \exp\Big(-\frac{i}{2}\vartheta(v)\,\sigma_y\Big) \in SU(2). \tag{159}$$

Lemma 5.7 (Explicit form of $U(v)$)**.** *Let $\vartheta = \vartheta(v)$. Then*

$$U(v) = \cos\Big(\frac{\vartheta}{2}\Big)\,I - i\sin\Big(\frac{\vartheta}{2}\Big)\,\sigma_y = \begin{pmatrix} \cos(\vartheta/2) & -\sin(\vartheta/2) \\ \sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix}. \tag{160}$$

In particular $U(v)$ is real orthogonal as a matrix and $U(v)^* = U(v)^{-1} = U(v)^\top$.

Proof. Since $\sigma_y^2 = I$, the exponential series splits into cosine and sine parts, yielding $U(v) = \cos(\vartheta/2)I - i\sin(\vartheta/2)\sigma_y$. Multiplying out with σ_y gives the displayed matrix.

5.3. PV-indexed microcausal observables

Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\mathcal{A} := B(\mathcal{H})$. Define commuting subalgebras

$$\mathcal{A}_A := B(\mathcal{H}_A) \otimes I, \qquad \mathcal{A}_B := I \otimes B(\mathcal{H}_B). \tag{161}$$

Planar settings and planar Pauli observable. For an angle $a \in \mathbb{R}/2\pi\mathbb{Z}$, define

$$\sigma(a) := \cos a\, \sigma_x + \sin a\, \sigma_z. \tag{162}$$

Then $\sigma(a)$ is self-adjoint and satisfies $\sigma(a)^2 = I$.

Lemma 5.8 (Conjugation by $U(v)$ shifts planar angles)**.** *Let $\vartheta = \vartheta(v)$. For every $a \in \mathbb{R}/2\pi\mathbb{Z}$,*

$$U(v)^*\, \sigma(a)\, U(v) = \sigma(a + \vartheta). \tag{163}$$

Proof. The unitary $U(v) = e^{-i\vartheta\sigma_y/2}$ implements a rotation about the y-axis. One computes

$$U^*\sigma_x U = \sigma_x \cos\vartheta + \sigma_z \sin\vartheta, \qquad U^*\sigma_z U = \sigma_z \cos\vartheta - \sigma_x \sin\vartheta, \tag{164}$$

hence

$$U^*(\cos a\, \sigma_x + \sin a\, \sigma_z)U = \sigma_x \cos(a+\vartheta) + \sigma_z \sin(a+\vartheta) = \sigma(a+\vartheta). \tag{165}$$

Definition 5.9 (PV-indexed microcausal ± 1 observables)**.** Define, for $a, b \in \mathbb{R}/2\pi\mathbb{Z}$,

$$A_v(a) := \big(U(v)^*\sigma(a)U(v)\big) \otimes I \ \in\ \mathcal{A}_A, \qquad B_v(b) := I \otimes \big(U(v)^*\sigma(b)U(v)\big) \ \in\ \mathcal{A}_B. \tag{166}$$

Lemma 5.10 (Microcausality and bounded outcomes)**.** *For all a, b and all $v \in (-c, c)$:*

(i) $[A_v(a), B_v(b)] = 0$*;*

(ii) $A_v(a)^* = A_v(a)$ *and* $B_v(b)^* = B_v(b)$*;*

(iii) $A_v(a)^2 = B_v(b)^2 = I \otimes I$*.*

Proof. (i) holds because $A_v(a) \in B(\mathcal{H}_A) \otimes I$ and $B_v(b) \in I \otimes B(\mathcal{H}_B)$ commute. (ii) and (iii) follow from self-adjointness and $\sigma(a)^2 = I$ plus invariance under unitary conjugation.

5.4. Singlet state, Pauli correlators, and v-invariance

Singlet vector state. Let $\{|0\rangle, |1\rangle\}$ be the standard basis of $\mathbb{C}^2$. Define

$$|\psi^-\rangle := \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) \in \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad \omega(X) := \langle\psi^-|X|\psi^-\rangle. \tag{167}$$

Lemma 5.11 (Singlet invariance under joint rotations)**.** *For every $U \in SU(2)$,*

$$(U \otimes U)|\psi^-\rangle = |\psi^-\rangle. \tag{168}$$

Consequently, for all 2×2 matrices X, Y,

$$\omega\big((U^*XU) \otimes (U^*YU)\big) = \omega(X \otimes Y). \tag{169}$$

Proof. The singlet spans the unique one-dimensional antisymmetric subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $U \otimes U$ preserves that subspace and has determinant 1 on it, hence acts as identity. The stated expectation identity follows by inserting $(U \otimes U)^*(U \otimes U) = I$ and using $(U \otimes U)|\psi^-\rangle = |\psi^-\rangle$.

Lemma 5.12 (Pauli correlators in the singlet state)**.** *For $i, j \in \{x, y, z\}$,*

$$\omega(\sigma_i \otimes \sigma_j) = -\delta_{ij}. \tag{170}$$

Proof. This is standard; one direct route is to use $(\sigma_i \otimes I)|\psi^-\rangle = -(I \otimes \sigma_i)|\psi^-\rangle$ and $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$.

Corollary 5.13 (Dot-product form for Pauli observables in the singlet state)**.** *For any real unit vectors $u, w \in \mathbb{R}^3$,*

$$\omega\big((\boldsymbol{\sigma} \cdot u) \otimes (\boldsymbol{\sigma} \cdot w)\big) = -u \cdot w. \tag{171}$$

Proof. Expand in the Pauli basis and apply Lemma 5.12.

Theorem 5.14 (PV-indexed microcausal model yields the singlet law unconditionally)**.** *Let $A_v(a), B_v(b)$ be the PV-indexed microcausal observables of Definition 5.9, and let ω be the singlet state above. Then for all $a, b \in \mathbb{R}/2\pi\mathbb{Z}$ and all $v \in (-c, c)$,*

$$E_v(a, b) := \omega\big(A_v(a)B_v(b)\big) = -\cos(a - b). \tag{172}$$

In particular, the correlator table is independent *of v even though v changes the operators by conjugation.*

Proof. By Definition 5.9,

$$A_v(a)B_v(b) = \big(U^*\sigma(a)U\big) \otimes \big(U^*\sigma(b)U\big), \qquad U = U(v). \tag{173}$$

By Lemma 5.8, $U^*\sigma(a)U = \sigma(a+\vartheta)$ and $U^*\sigma(b)U = \sigma(b+\vartheta)$. Identify $\sigma(\alpha) = \boldsymbol{\sigma}\cdot u(\alpha)$ with

$$u(\alpha) := (\cos\alpha, 0, \sin\alpha) \in \mathbb{R}^3. \tag{174}$$

Then Corollary 5.13 gives

$$E_v(a,b) = -u(a+\vartheta)\cdot u(b+\vartheta) = -\cos\big((a+\vartheta)-(b+\vartheta)\big) = -\cos(a-b). \tag{175}$$

(Equivalently, by Lemma 5.11, the joint conjugation by $U(v)$ cancels directly in the singlet state.)

5.5. Tsirelson value on a standard quartet

Corollary 5.15 (Tsirelson value for the PV-indexed microcausal model)**.** *Choose the standard Tsirelson quartet*

$$a_0 = 0, \qquad a_1 = \frac{\pi}{2}, \qquad b_0 = \frac{\pi}{4}, \qquad b_1 = -\frac{\pi}{4}. \tag{176}$$

Then for every $v \in (-c, c)$*,*

$$S_v := \big|E_v(a_0,b_0) + E_v(a_0,b_1) + E_v(a_1,b_0) - E_v(a_1,b_1)\big| = 2\sqrt{2}. \tag{177}$$

Proof. By Theorem 5.14, $E_v(a,b) = -\cos(a-b)$. Substitution yields $2\sqrt{2}$.

6. Discussion and Conclusions

6.1. Summary: PV, Bell–CHSH, selection, and microcausality

The results isolate, in theorem form, the exact logical status of "PV-to-Tsirelson" claims:

(C1) **Unconditional (rung-0) no-go.** If one remains in the classical Bell–CHSH hypothesis class (Kolmogorov probability, MI, Bell locality, bounded outcomes, and unconditional single-ensemble correlators), then CHSH$\leq$ 2 on every quartet (Theorem 2.5); thus Tsirelson-scale $-\cos$ tables are impossible on a Tsirelson quartet (Proposition 4.17).

(C2) **Exceptional-locus semantics either cancels or becomes selection.** Strict manipulation of the PV radical equation cancels the PV factor and yields a compatibility locus independent of v (Proposition 3.3); the CAS ratio collapses to $\pm c$ off-locus and is undefined on-locus (Proposition 3.5). If exceptional-locus/defined-only semantics changes which trials contribute to correlations, it induces an acceptance map and a family $\{\nu_{ab}\}$ of setting-indexed accepted laws (Proposition 3.7), i.e. rung-1 semantics.

(C3) **Rung-1 (selection) realization is explicit and exact.** A Bell-local deterministic base model together with a setting-dependent acceptance rule can reproduce the full singlet *accepted-sample* law (unbiased marginals and correlator $-\cos(a-b)$) exactly (Theorem 4.7, Lemma 4.12, Corollary 4.13). This does not contradict Bell because the CHSH correlators are evaluated under different accepted laws. In the explicit construction, the acceptance mechanism is intrinsically joint/coincidence-style: it is not locally factorizable as $\gamma_A(a,\lambda)\gamma_B(b,\lambda)$ (Proposition 4.6), and its acceptance-rate profile is explicitly computable (Proposition 4.9).

(C4) **Rung-2 (microcausal) realization is unconditional and exact.** PV can parameterize a unitary conjugation of local observables inside commuting Alice/Bob subalgebras, and in the singlet state the resulting unconditional correlator is exactly $-\cos(a-b)$ (Theorem 5.14), attaining $2\sqrt{2}$ on a standard quartet (Corollary 5.15). This route lies outside Bell's Kolmogorov hypothesis class because the underlying probability is noncommutative.

6.2. A quantitative rung-1 diagnostic: dispersion constraints

Whenever a Tsirelson-scale value is attributed to rung-1 (selection) semantics under Bell locality and MI at emission, the quartet dispersion Δ_Q must be large: $S_{\text{obs}} = 2\sqrt{2}$ implies $\Delta_Q \geq \sqrt{2}-1$ (Corollary 2.22). Thus the selection route carries an explicit quantitative burden.

7. Phenomenological Velocities Derived from Bell–CHSH Selection Geometry

7.1. Notation (Bell–CHSH under setting-dependent acceptance) Let $\rho \in \mathcal{P}(\Lambda)$ be a measurement-independent (MI) prior on hidden variables $\lambda \in \Lambda$. Let $a \in S_A$, $b \in S_B$ be settings and let

$$A : S_A \times \Lambda \to [-1,1], \qquad B : S_B \times \Lambda \to [-1,1] \tag{178}$$

be Bell-local response functions.

Selection is modeled by an acceptance rule (acceptance probability)

$$\gamma : S_A \times S_B \times \Lambda \to [0,1]. \tag{179}$$

Define the acceptance rate

$$Z(a,b) := \int_\Lambda \gamma(a,b,\lambda)\, d\rho(\lambda) \in (0,1], \tag{180}$$

and the accepted hidden-variable law

$$\nu_{ab}(E) := \frac{1}{Z(a,b)} \int_E \gamma(a,b,\lambda)\, d\rho(\lambda), \qquad E \in \mathcal{F}. \tag{181}$$

The accepted-sample correlator is

$$E_{\text{obs}}(a,b) := \int_\Lambda A(a,\lambda)B(b,\lambda)\, d\nu_{ab}(\lambda). \tag{182}$$

Fix a CHSH quartet $Q = \{(a_i, b_j) : i,j \in \{0,1\}\}$. Write

$$\nu_{ij} := \nu_{a_i b_j}, \qquad E_{ij} := E_{\text{obs}}(a_i, b_j), \qquad S_{\text{obs}} := \big|E_{00} + E_{01} + E_{10} - E_{11}\big|. \tag{183}$$

Let $\text{TV}(\cdot,\cdot)$ denote total variation distance on $\mathcal{P}(\Lambda)$.

Define the quartet dispersion and diameter:

$$\Delta_Q := \inf_{\mu \in \mathcal{P}(\Lambda)} \Big(\text{TV}(\nu_{00},\mu) + \text{TV}(\nu_{01},\mu) + \text{TV}(\nu_{10},\mu) + \text{TV}(\nu_{11},\mu)\Big), \tag{184}$$

$$D_Q := \max_{(i,j)\neq(k,\ell)} \text{TV}(\nu_{ij}, \nu_{k\ell}). \tag{185}$$

A sharp universal inflation bound is

$$S_{\text{obs}} \le 2 + 2\Delta_Q \le \min\{4,\ 2 + 6D_Q\}. \tag{186}$$

Throughout this section, $c > 0$ is a fixed velocity scale (used only to normalize PV parameters into $[0,c]$).

7.2. PV Form I: CHSH-inflation PV from dispersion From (186),

$$S_{\text{obs}} \le 2 + 2\Delta_Q \quad \Longrightarrow \quad \Delta_Q \ge \frac{S_{\text{obs}} - 2}{2}. \tag{187}$$

Define the *dispersion PV* (capped to $[0,c]$):

$$\boxed{v_\Delta(Q) := c\, \min\{1, \Delta_Q\} \in [0,c].} \tag{188}$$

Then

$$S_{\text{obs}} \le 2 + 2\frac{v_\Delta(Q)}{c}. \tag{189}$$

A *minimum required PV* to reproduce a given S_{obs} (in a Bell-local MI selection-based explanation) is

$$\boxed{v_\Delta^{\min}(Q) = c\, \frac{S_{\text{obs}} - 2}{2}.} \tag{190}$$

Tsirelson scale. If $S_{\text{obs}} = 2\sqrt{2}$, then

$$v_\Delta^{\min}(Q) = c(\sqrt{2} - 1) \approx 0.4142\, c. \tag{191}$$

7.3. PV Form II: diameter PV from pairwise TV geometry From (186),

$$S_{\text{obs}} \le 2 + 6D_Q \quad \Longrightarrow \quad D_Q \ge \frac{S_{\text{obs}} - 2}{6}. \tag{192}$$

Define the *diameter PV*

$$\boxed{v_D(Q) := c\, D_Q \in [0, c].} \tag{193}$$

Then

$$S_{\text{obs}} \le 2 + 6\frac{v_D(Q)}{c}. \tag{194}$$

Hence the minimum required v_D for a given S_{obs} is

$$\boxed{v_D^{\min}(Q) = c\,\frac{S_{\text{obs}} - 2}{6}.} \tag{195}$$

Tsirelson scale. If $S_{\text{obs}} = 2\sqrt{2}$, then

$$v_D^{\min}(Q) = c\,\frac{\sqrt{2} - 1}{3} \approx 0.1381\, c. \tag{196}$$

Rescaled variant matching the dispersion coefficient. Since $\Delta_Q \le 3D_Q$, define also

$$v_{3D}(Q) := c\, \min\{1, 3D_Q\}, \tag{197}$$

so that $S_{\text{obs}} \le 2 + 2\, v_{3D}(Q)/c$.

7.4. PV Form III: acceptance-rate PV via a Lorentz-factor identification A Lorentz-like factor is $\kappa(v) := \sqrt{1 - v^2/c^2} \in [0, 1]$. A natural Bell-selection analogue is the acceptance rate $Z(a, b) \in (0, 1]$. Define

$$\boxed{\kappa_{ab} := Z(a, b), \qquad v_Z(a, b) := c\sqrt{1 - Z(a, b)^2} \in [0, c].} \tag{198}$$

Thus $Z = 1$ implies $v_Z = 0$, and $Z \downarrow 0$ implies $v_Z \uparrow c$.

Coarse link between Z and fair-sampling deviation. One has the (coarse) bound

$$\mathrm{TV}(\nu_{ab}, \rho) \le 1 - Z(a, b). \tag{199}$$

If one requires (as a necessary condition for Tsirelson-scale CHSH under selection) that for at least one setting pair (a, b) in the quartet,

$$\mathrm{TV}(\nu_{ab}, \rho) \ge \frac{\sqrt{2} - 1}{4}, \tag{200}$$

then

$$1 - Z(a,b) \geq \frac{\sqrt{2}-1}{4} \quad \Longrightarrow \quad Z(a,b) \leq 1 - \frac{\sqrt{2}-1}{4} = \frac{5-\sqrt{2}}{4} \approx 0.8964. \tag{201}$$

Consequently,

$$\boxed{\frac{v_Z(a,b)}{c} \geq \sqrt{1 - \left(\frac{5-\sqrt{2}}{4}\right)^2} = \frac{1}{4}\sqrt{10\sqrt{2}-11} \approx 0.443.} \tag{202}$$

Example (explicit $-\cos$ **selection simulation on the Tsirelson quartet).** For the standard Tsirelson quartet separations $\Delta = \pi/4$ and $3\pi/4$, the explicit construction yields

$$Z = Z(\pi/4) = 3 - \frac{3\sqrt{2}}{2} \approx 0.878679656, \tag{203}$$

hence

$$\boxed{\frac{v_Z}{c} = \sqrt{1 - \left(3 - \frac{3\sqrt{2}}{2}\right)^2} = \sqrt{\frac{18\sqrt{2}-25}{2}} \approx 0.477.} \tag{204}$$

7.5. PV Form IV: a Lorentz–rapidity PV from two-bin ($AB = \pm 1$) selection Fix a setting pair (a,b) and assume $f := AB \in \{\pm 1\}$. Define the unconditional correlator

$$E_{\text{full}} := \mathbb{E}_\rho[f] \in [-1,1], \qquad q_+ := \Pr_\rho(f=+1) = \frac{1+E_{\text{full}}}{2}, \qquad q_- := \Pr_\rho(f=-1) = \frac{1-E_{\text{full}}}{2}. \tag{205}$$

Assume acceptance depends only on f:

$$\Pr(\text{Acc} \mid f=+1) = \eta_+, \qquad \Pr(\text{Acc} \mid f=-1) = \eta_-, \qquad \eta_\pm \in (0,1], \tag{206}$$

and define the odds ratio

$$r := \frac{\eta_+}{\eta_-} \in (0,\infty). \tag{207}$$

Then the accepted correlator is

$$E_{\text{obs}} = \mathbb{E}[f \mid \text{Acc}] = \frac{\eta_+ q_+ - \eta_- q_-}{\eta_+ q_+ + \eta_- q_-} = \frac{r q_+ - q_-}{r q_+ + q_-}. \tag{208}$$

Solving for r in terms of $(E_{\text{obs}}, E_{\text{full}})$ yields

$$r = \frac{q_-(1+E_{\text{obs}})}{q_+(1-E_{\text{obs}})} = \frac{(1-E_{\text{full}})(1+E_{\text{obs}})}{(1+E_{\text{full}})(1-E_{\text{obs}})}. \tag{209}$$

Define the correlation rapidity

$$u(E) := \operatorname{artanh}(E) = \frac{1}{2}\log\frac{1+E}{1-E}. \tag{210}$$

Then

$$\log r = 2u(E_{\text{obs}}) - 2u(E_{\text{full}}). \tag{211}$$

Define the rapidity shift

$$\xi := \frac{1}{2}\log r = u(E_{\text{obs}}) - u(E_{\text{full}}), \tag{212}$$

and define the *boost PV* by

$$\boxed{\frac{v_{\text{boost}}}{c} := \tanh\xi = \tanh\big(u(E_{\text{obs}}) - u(E_{\text{full}})\big).} \tag{213}$$

Using $\tanh(u(E)) = E$ and $\tanh(A-B) = \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$, one obtains the closed form

$$\boxed{\frac{v_{\text{boost}}}{c} = \frac{E_{\text{obs}} - E_{\text{full}}}{1 - E_{\text{obs}}E_{\text{full}}} \in (-1, 1).} \tag{214}$$

Application to the sign–cos base model and $-\cos$ **acceptance.** For the sign–cos base model,

$$E_{\text{full}}(\Delta) = -1 + \frac{2\Delta}{\pi}, \qquad E_{\text{obs}}(\Delta) = -\cos\Delta, \qquad \Delta \in [0, \pi], \tag{215}$$

so

$$\boxed{\frac{v_{\text{boost}}(\Delta)}{c} = \frac{-\cos\Delta - \left(-1 + \frac{2\Delta}{\pi}\right)}{1 - (-\cos\Delta)\left(-1 + \frac{2\Delta}{\pi}\right)} = \frac{1 - \frac{2\Delta}{\pi} - \cos\Delta}{1 - \cos\Delta\left(1 - \frac{2\Delta}{\pi}\right)}.} \tag{216}$$

On the Tsirelson separations:

$$\Delta = \frac{\pi}{4}: \quad \frac{v_{\text{boost}}}{c} = \frac{-\frac{\sqrt{2}}{2} - \left(-\frac{1}{2}\right)}{1 - \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)} = \frac{2 - 3\sqrt{2}}{7} \approx -0.320377, \tag{217}$$

$$\Delta = \frac{3\pi}{4}: \quad \frac{v_{\text{boost}}}{c} = \frac{+\frac{\sqrt{2}}{2} - \left(+\frac{1}{2}\right)}{1 - \left(+\frac{\sqrt{2}}{2}\right)\left(+\frac{1}{2}\right)} = \frac{3\sqrt{2} - 2}{7} \approx +0.320377. \tag{218}$$

7.6. PV Form V: tag-auditable PVs (Lane B) Let $X : \Lambda \to \mathsf{X}$ be a measurable tag and write $\nu_{ab}^X := \nu_{ab} \circ X^{-1}$. For a quartet Q, define the resolved diameter and dispersion:

$$D_{Q,X} := \max_{q \neq q'} \mathrm{TV}(\nu_q^X, \nu_{q'}^X), \qquad \Delta_{Q,X} := \inf_{\mu^X \in \mathcal{P}(\mathsf{X})} \sum_{q \in Q} \mathrm{TV}(\nu_q^X, \mu^X). \tag{219}$$

Then $D_{Q,X} \le D_Q$ and $\Delta_{Q,X} \le \Delta_Q$ by data processing.

Define tag-auditable PVs:

$$\boxed{v_{\Delta,X}(Q) := c\,\min\{1, \Delta_{Q,X}\}, \qquad v_{D,X}(Q) := c\,D_{Q,X}.} \tag{220}$$

If X is sufficient for the accepted family $\{\nu_{ab}\}$, then $\Delta_{Q,X} = \Delta_Q$ and $D_{Q,X} = D_Q$, so these PVs become exact.

7.7. PV Form VI (rung-2 / microcausal): noncommutativity PV from commutators In a microcausal C^*-probability model with ± 1 observables A_0, A_1 (Alice) and B_0, B_1 (Bob), define the Bell operator

$$\mathcal{B} := A_0(B_0 + B_1) + A_1(B_0 - B_1). \tag{221}$$

One has the identity

$$\mathcal{B}^2 = 4I - [A_0, A_1][B_0, B_1], \tag{222}$$

hence

$$\|\mathcal{B}\|^2 = \|\mathcal{B}^2\| \le 4 + \|[A_0, A_1]\| \, \|[B_0, B_1]\|. \tag{223}$$

Define the normalized commutator magnitudes

$$\alpha := \frac{\|[A_0, A_1]\|}{2} \in [0, 1], \qquad \beta := \frac{\|[B_0, B_1]\|}{2} \in [0, 1], \tag{224}$$

and define the *noncommutativity PV*

$$\boxed{\frac{v_{\mathrm{nc}}}{c} := \sqrt{\alpha\beta} = \sqrt{\frac{\|[A_0, A_1]\| \, \|[B_0, B_1]\|}{4}} \in [0, 1].} \tag{225}$$

Then the CHSH value $S_\omega := |\omega(\mathcal{B})|$ satisfies

$$\boxed{S_\omega \le \|\mathcal{B}\| \le 2\sqrt{1 + \left(\frac{v_{\mathrm{nc}}}{c}\right)^2} \le 2\sqrt{2}.} \tag{226}$$

7.8. Summary The Bell–CHSH selection framework induces multiple natural PV parameters:

- *Quartet (selection-geometry) PVs:* $v_\Delta(Q)$ from Δ_Q and $v_D(Q)$ from D_Q.
- *Acceptance-factor PV:* $v_Z(a, b)$ from $Z(a, b)$ by the identification $Z = \sqrt{1 - v^2/c^2}$.
- *Lorentz–rapidity PV:* $v_{\mathrm{boost}}/c = (E_{\mathrm{obs}} - E_{\mathrm{full}})/(1 - E_{\mathrm{obs}}E_{\mathrm{full}})$ for two-bin ($AB = \pm 1$) selection, with rapidity shift $\xi = \operatorname{artanh}(E_{\mathrm{obs}}) - \operatorname{artanh}(E_{\mathrm{full}})$.
- *Tag-auditable PVs:* $v_{\Delta,X}(Q)$ and $v_{D,X}(Q)$ from resolved dispersions $\Delta_{Q,X}, D_{Q,X}$.
- *Microcausal/noncommutative PV:* v_{nc} from commutator magnitudes, yielding $S \le 2\sqrt{1 + (v_{\mathrm{nc}}/c)^2}$.

7.9. Closing statement Bell–CHSH is a theorem about unconditional expectations in a single commutative probability model. PV-style exceptional-locus semantics cannot change that theorem unless it changes the ensemble (selection) or leaves the commutative model class.

Data Availability The code and data that support the findings of this study are available at https://zenodo.org/records/18728172 (https://doi.org/10.5281/zenodo.18728172).

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