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Original Paper

Topological Structures to account for Quantum Questions

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Abstract: In this paper I inquire into the applicability of topological structures in the mathematical modeling of certain quantum situations and attempt an interpretation, on the level of metatheory, of phenomena associated with the time evolution of quantum processes and the individuality of quantum objects upon observation. Accordingly the paper engages, on the one hand, in an epistemologically oriented discussion of the merits of topological approaches concerning natural science in general and certain questions of quantum theory in particular, and on the other, in an elaboration of a proper topological structure to deal with the mathematical aspects of an open question of the theory of quantum histories, the latter as developed mainly by C. J. Isham and co. On this motivation a brief discussion concerning the topological nature of the Bohm-Aharonov effect is thought to be in order. Overall the primary focus is to discuss the relevance of topology, as a pure mathematical theory of structures, with the quantum context and in particular of the quantum histories processes over temporal points and continuous time intervals.

Keywords: Continuous-time proposition; individuality; openness; quantum history; single-time proposition; temporal point; temporal support; topological structure

1. Introduction

Given that, on the one hand, Hilbert spaces, operators and tensor theories and (relatively lately) topos theory, etc. play a key role in the formal-mathematical elaboration of quantum theory and, on the other, that quantum state spaces have structural properties expressible in terms of the structure itself, it is natural that topology has acquired an increasing relevance with quantum theory. Indeed, there is a remarkable literature on the application of topological or topologically related theories to aspects of quantum theory. Examples include optics for which topology enters either via the study of non-dynamical phases or in the construction of optical systems that can simulate the behavior of peculiar solid-state systems known as topological insulators, cohomology theory as used in physics to compute the topological structure of gauge fields, Schlesinger's application of aspects of topos theory toward the quantization of the whole category of topological spaces and continuous injective maps, [35], and the application of topological aspects of category theory to quantum theory in [12]. Another example is the connection of knot theory and topological braiding with quantum entanglement which could be a way to understand the properties of entangling gates in topological quantum computing, [23]. The latter work proceeds (in its updated version) to show how quantum entanglement and topological connectivity can be intimately related ([23]; updated version). It must be pointed out that it is generally acknowledged it is due to the progress of quantum mechanics that topology has gained a prominent role in theoretical physics. Dirac's analysis of magnetic monopoles and the discovery of the Bohm-Aharonov effect are primary instances of the relevance of topological ideas with the quantum mechanical framework (e.g., in the emergence of quantization outcomes) and together with the quantum Hall effect are key references on the matter. The Bohm-Aharonov effect, in particular, is briefly discussed for its 'pedagogical' merits and in line with the scope of this article in section 3.

Some theoretical elaboration by applying topological notions has been done in quantum temporal histories by C. J. Isham, mainly in [20] and [22], but with no particular success toward achieving real breakthroughs in quantum theory in its own merit. Worthy of mention along these lines is Hawkins' *et al* and Markopoulou's work on quantum causal histories which can be treated either as algebraic or 'directed' topological quantum field theories respectively in [15] and [29]. Yet this relatively limited activity in applying topology to quantum theory, more accurately to quantum histories theory, should not be discouraging from pursuing an epistemological overview of the applicability of topological ideas to quantum theory and the interpretability of the latter in terms of the former. This is especially true in view of the fact that there is so far a very limited literature on the insights topological ideas may bring to concrete quantum-theoretical questions and furthermore on the epistemological and possibly ontological interpretation one might give to these insights.

This is the kind of undertaking taken up in this paper. First in section 2, I generally discuss the role of topological concepts in quantum theory and to a limited extent in natural science. In section 3, I discuss the Bohm-Aharonov effect with the purpose of highlighting the epistemological aspect of topological notions applied to quantum situations. In section 4, I discuss the formal and epistemological relevance of topological openness in the description of quantum processes, and the 'ontological' status of temporal points. The discussion bears also on the possibility of interpreting single time-points in quantum history processes in terms of topological openness. This serves as a motivation for dealing with a question on the formal treatment of quantum histories in section 5 which is of a more technical content in presenting a tentative topological solution, one that actually fails in non-Abelian sublattices, to a question raised in [3], namely of determining a continuous map from the lattice of single-time propositions to the lattice of continuous-time ones.

In this sense section 5 may find its relevance with the more philosophically-epistemologically oriented rest of the paper insofar as it brings up the ways topological notions and methods may influence our conception of temporal points vs temporal continuum in a quantum context while pointing at the same time to their inherent limits as formal constructs.

2. The relevance of topology with questions in natural science

A notable mathematical activity on a conception of structures modelizing phenomena in natural sciences, such as network organization, cellular evolution, system interactions, etc., has been seen in the last years that views structures in terms of their explanatory capacities as such rather than through the mathematics describing them. In this sense structural explanations is what makes the underlying mathematics meaningful in that "the mathematics not only represent the mechanisms' settings and functioning, they also explain why a set of mechanisms is constrained in a specific way, necessarily yielding a range of outcomes that possess a given property." ([17], p. 120). Huneman has made the tantalizing argument that the mathematics of a formal metatheory is the reason why some macroscopic systems are exhibiting a regularity in their existence, and further that the interaction with their environment is a mathematical, more specifically in case we talk about structural properties, a topological fact. In this sense a topological property is instantiated,

"[..] by all mechanisms in the considered systems, but it's only in virtue of the fact that they instantiate this property that those are themselves explanatory of anything. Topology being about invariance through a class of continuous transformations, topological explanations are explanations in which the possibility and impossibility of some systems to reach some sets of final states or behaviors is explained by the topological fact which they instantiate, specifying which states are topologically equivalent and which are not, hence are not likely to be reached by the system." (ibid., p. 120).

A parallel view is brought up by Clifton in which he claims that "if we are to understand quantum theory, in which even talk of indeterministic causal processes breaks down, we will have to take seriously the idea that locating phenomena within a coherent and unified mathematical model is explanatory in itself." ([8], p. 6). Even though Clifton does not specifically refer to topological notions, his motivation is *in rem* related to these concepts inasmuch as the intended program is to take hold of the intrinsic structure of relativistic quantum field theory by associating algebras of local observables with regions of space-time (ibid., p. 18).

As a matter of fact a key difference between mechanistic or semantic approaches and topological ones is that the latter are interpretable in terms of the topological properties of the structure itself, e.g., topological properties of systems such as compactness and connectedness remain invariant under continuous transformations and thus are unconstrained by local factors or even by causal relations depending on the scale of each particular instance under consideration. Topological explanations do not presuppose in principle a causal connection in terms of realization at a local scale since the concept that basically underlies topological realization[1] is connectivity in the classical topological sense, namely connectivity based on the notion of openness as remaining invariable across homeomorphisms (i.e., one-to-one continuous transformations in both straight and reverse sense). Consequently to the extent that different patterns of connectivity correspond to different topologies, this implies distinct realization bases which is by itself explanatory in this sense without the need to take account of micro-scale factors or of the realization relation as a functional relation between a concept and a micro-physical description of its causal or functional roles conditioned on possibly different theories of meaning. This way even though the explanatory relation stands between a physical fact or property and a topological property, in a topological explanation one should take account of explanatory terms based on the mathematical references of topologies rather than the "actual ontic details of various systems" ([24], p. 96). A special case may be the way a topological theory deals with individual points of a structure (zero-elements or singletons), a fact reflected at the level of quantum theory in the inconveniences arising in the construction of continuous transformations of evolution operators defined at time instants. From a certain viewpoint there might be a certain 'affinity' between the ways the notion of a time-instant shapes quantum-mechanical concepts, e.g., those of disentanglement and quantum measurement, and the way points may be interpreted as non-vacuous, yet 'open' entities depending on the kind of topology. To the extent that time in quantum theory is generally considered as an external parameter of the system it is reasonable to assume that a time instant loses its classical sharp individual sense and may instead acquire an 'inner' content possible to influence the phenomenology of the situation.

In a reassessment of the traditional view that scientific explanations are necessarily based on physical or causal models of phenomena in favor of the explanatory power of the mathematical models themselves as used in quantum mechanics, Dorato and Felline have proposed formal properties of mathematical models that represent certain features of

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[1] The topological realization is defined as follows: The realization relation holds between a topology $\mathcal{T}$ and a system $\mathcal{K}$ on the condition that the system $\mathcal{K}$ realizes topology $\mathcal{T}$ whenever the elements of $\mathcal{K}$ are interconnected in the mode of connectivity implied by the topology $\mathcal{T}$.

well-known quantum processes in a way that these physical properties are made intelligible. More specifically, Heisenberg's uncertainty relation $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$ between position and momentum is understood as a straightforward consequence of the formal-mathematical properties of the Fourier transform in the Hilbert space modelization of the quantum system in question ([10], p. 6). On this account and by virtue of the reduction of quantum non-locality phenomena to non-factorizable, entangled states of tensor products of Hilbert spaces as formal representatives,[2] they have put the claim "that structural explanations provide a common ground for understanding the explanandum (*auth. note*: the physical system) in question, independently of the various different ontologies underlying the different interpretations of quantum theory" (ibid., p. 7). Therefore one may say that the existence of structure-preserving morphisms from the Hilbert space associated with the physical system to the physical properties (or relations) meant as observables of the system makes that properties of the physical model can be made intelligible by the formal properties of the mathematical model. Accordingly in Dorato's & Felline's view physical regularities or irregularities may be explained or understood in terms of the mathematical and, in particular, topological facts and not the other way around.

Yet the metatheoretical question of why the topological and more generally the formal-mathematical reduction of processes in a quantum context may be provided with an explanatory power of a possibly generic character should be the object of a further discussion going perhaps as far as the turf of ontological philosophy. However at this point one should limit himself within the bounds of the epistemological context and possibly within the perspective of scientific or structural realism.

3. The case of the Bohm-Aharonov effect

To highlight the necessity of assumption of a connected topological texture of the phase space in a well-known phenomenon of quantum mechanics, I will enter into a brief discussion of the Bohm-Aharonov effect. As known this effect is associated with the topological structure of the classical configuration space in terms of which

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[2] Entangled quantum states can structurally explain non-locality phenomena essentially by the following two facts:

(i) The Hilbert model validates the principle of superposition, that is, the sum of vectors of the Hilbert space (physical states) is also a vector of that space (a physically possible state) and

(ii) some superpositions of state vectors in the tensor products of the Hilbert spaces associated to subsystems of a composite system cannot be expressed as a tensor product of any of the state vectors corresponding to the component Hilbert spaces. This is a key feature of the mathematical-structural explanation as by this formal non-factorizability is explained the entanglement or non-locality of quantum phenomena ([10], p. 13).

the wave function must be described. In this connection the implications of quantum interactions may be encoded in the topology of the configuration space in an approach that implies it is the points of configuration space in the particular topology that confer individuality and re-identifiability of quantum objects while instantiating by the same token their properties. Perhaps there should be a deeper connection on a phenomenological, subjectively founded level between quantum objects as objects of a physicalistic language and their representation as formal-ontological ones embeddable in the domain of a formal structure. But this is a question to be thoroughly treated for its own sake and this is left perhaps for a future article.

My reference to certain key features of the Bohm-Aharonov effect serves mainly to point to the way certain peculiarities on the observational-physical level may be translated by mathematical formalization into peculiar topological properties of the configuration space. This has also to do with the kind of gauge invariance interpretation one chooses to assign to the phase space of the electromagnetic field so as to accommodate the peculiar features of the Bohm-Aharonov effect, an approach that by itself points through the notion of holonomies to the specific topological structure of the phase space and the non-locality characteristics that go with its global character ([6], pp. 544, 550). The particular topic may merit a more extensive and thorough discussion but at this point I only touch on the issue to the extent that it connects with my overall view of the question of the limits of 'observation' and its reflection in the corresponding topological structure.[3]

The peculiarity of the Bohm-Aharonov effect, roughly put, has to do with the presence of a solenoid that causes a shift in the interference pattern of a double slit in the notable absence of an external magnetic field. As a matter of fact the physical effect observed, which is the change in the phase difference of the electron interference pattern $\Delta\delta = \frac{e}{\hbar}\int \mathrm{curl}\mathbf{A}\cdot\mathbf{dS}$, (e the electron charge), depends only on $\mathrm{curl}\mathbf{A}$[4] in a way that it can be deduced that an electron is influenced by (magnetic) fields which are only non-zero in regions inaccessible to it. In formal terms, this amounts to a non-locality of the integral $\oint \mathbf{A}\cdot\mathbf{dr}$ ([32], pp. 100-101).

A a matter of fact the Bohm-Aharonov effect is due to the non-trivial topology of the vacuum and the fact that electrodynamics is a gauge theory. It has been demonstrated though that the vacuum in gauge theories has a rich mathematical structure associated with certain physical consequences and the Bohm-Aharonov effect is a major illustration of the deeper connection that may exist between certain irregular kinds of physical interaction and the corresponding 'pathologies' in topological modelization.

Indeed the Bohm-Aharonov effect is formally reduced to a certain topological

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[3] Relative expository or research work can be found in [1], [2], [6] and [30].

[4] The vector potential $\mathbf{A}$ is linked to the magnetic induction $\mathbf{B}$ by the well-known formula $\mathbf{B} = \mathrm{curl}\mathbf{A}$.

peculiarity of the configuration space of the null-field. This is a plane with a hole in it which is the non-simply connected circle S^1. In mathematical formalization, this generates a many-valued gauge function[5] x mapping the group space S^1 onto the configuration space of the experiment $S^1 \times R$ such that not all such x are deformable to a constant gauge function. In that case, it would be generated $A_\mu = 0$ and no Bohm-Aharonov effect (ibid., p. 105). Mathematically the function x satisfying $A = \nabla x$ turns out to be a many-valued function and this becomes possible because the space on which it is defined is non-simply connected. Put in other words, the group space of the gauge group of electromagnetism $U(1)$[6] is the non-simply connected circle S^1 where, intuitively speaking, a non-simply connected space is one in which not all curves may be continuously shrunk to a point.

If x was single-valued, then $B = \text{curl}A = \text{curl}\nabla x \equiv 0$ everywhere, so there would be no magnetic flux Φ and consequently no physical effect taking into account that $\Delta\delta = \frac{e}{\hbar}\Phi$.[7]

In view of the discussion so far we take note of a recalibration of the irregular characteristics of the quantum effect in question to a peculiarity of the topology of the configuration space of the experiment that reduces, in the present case, to the non-simple connectedness of the topological structure of the group space S^1. In other words one may view the solenoid as a hole in the space of allowed field configurations in which the quantization arises from the topological fact that curves in A-space that enclose the solenoid are non-contractible. The integer n in the situation counts the number of times the loop encloses the singularity. It is a winding number characterizing the distinct homotopy classes of the field ([33], p. 16).

I note that, on a fundamental level, prior to the assumption of topological discontinuities, e.g., of the kind of the non-simple connectedness on the matter, one must assume a topological continuum further reducible in metatheoretical, in fact subjectively founded terms, to a notion of intuitive continuum possibly conceived, in turn, as the constancy across time of the flux of inner time consciousness. In phenomenological terms inner time consciousness may be seen as bridging in effect the context of an experimental preparation with that of the measurement as implemented by a conscious observer. Of course this may

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[5] There is generally nothing wrong with that the gauge function is multi-valued in the case a configuration space Q is non-simply connected as this is derived by topological considerations over the (unique up to isomorphism) universal covering space $\overline{Q}$ of Q. See [30], pp. 117-124.

[6] The first homotopy group of $U(1)$ is proved to be isomorphic to the group Z of integers under addition (the integer n corresponding to a closed path going around n times the circle S^1 in group space). This implies that the gauge group of electromagnetism $U(1)$ is non-simply connected thus making possible the Bohm-Aharonov effect. For details see: [32], pp. 103-104.

[7] In their exposition of an experiment in [7], Bohm and Hiley propose an interpretation which implies a physical effect for the vector potential A on the quantum level by means of a mathematical formalism which reduces again to a peculiarity of the mathematical model of the configuration space (ibid., pp. 50-54.)

further motivate a philosophically oriented discussion which is however beyond the scope of this article.[8]

4. Why topological openness of time points may be relevant with quantum processes

As time may be considered a co-constituting factor of each quantum situation it is reasonable to assume, from an epistemological standpoint, that a time instant may lose its classical sharp individuality sense and acquire instead an 'inner' content corresponding to an ongoing state of objectification in terms of a quantum measurement. In other words in the disentanglement of a quantum state-of-affairs upon detection temporal points associated with concrete detection instants acquire a 'fluidity' translated to the continuous evolution of the quantum state vector.

The question is in a certain sense linked with von Neumann's Projection Postulate (or 'the reduction of the wave function' postulate)[9] which assigns to the mathematical translation $\mathcal{S}(s(t))$ of the physical state $s(t)$ of a quantum system Q_i upon a first-kind measurement at time t the same eigenvector ψ_κ as to the translation of the state $s(t_1)$ of the quantum system Q_i at time t_1 soon after the measurement. As a matter of fact even if we assume von Neumann's Projection Postulate, or van Fraassen's modal interpretation of quantum mechanics as 'external' metatheoretical conditions, in a purely logical way we cannot be led by any analytical linguistic means to a complete description of the change of states, let alone the determination of the quantum state-in-transition at a 'sharp' time instant, that occurs during the measurement process in the compound system 'system+apparatus+observer' ([28], p. 164). This apparent non-eliminability of the temporal factor in quantum processes has led to a host of contrasting views concerning the epistemology of the situation ranging from pragmatic or scientific realism to subjectively founded and even to phenomenologically motivated interpretations. The core matter is that the notion of a time instant is radically different in quantum processes than in the classical context insofar as, in the former case, it may be considered as conditioned on a subjective constitution of a quantum state-of-affairs out of an agglomeration of potentialities in terms of the temporal-constitutive modes of the subjective factor. These constitutive modes, at least from a phenomenological viewpoint, may not be reducible to discrete, sharp temporal instants. This means that while in a classical context we may have a complete description of a physical process in terms of sharp, fully distinct temporal instants, in a quantum context, insofar as the measuring system (conscious 'observer' + measuring apparatus) may be considered as co-constitutive of a quantum state-of-affairs the notion

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8 The reader may find clues on these matters in [28], subsec. 3.2.

9 Historically von Neumann's Projection Postulate was preceded by Dirac's analogous VII postulate. See: [9], p. 36.

of a sharp time instant loses its classical meaning. In fact, a time instant becomes itself a procedural phase in the process of turning from potentialities to actualities, from a non-Boolean contextual to a Boolean decontextualized frame; think for instance the case of quantum disentanglements. Furthermore on the assumption of the co-constitutive role of the consciousness of an 'observer', our conception of quantum temporal instants cannot leave unaccountable the way temporal instants and generally temporality are constituted within human subjectivity, especially if the philosophical inclination one might have is such that a prominent role would be attributed to the absolute subjective factor, name it, e.g., the Husserlian phenomenological ego, the Heideggerian Dasein, etc. In the Husserlian phenomenological doctrine, for instance, there is no sharp temporal instant but, roughly said, a 'specious' present associating in a non-reductionistic (in fact an a priori) fashion an original impression with immediate past and future.[10] In this respect one might well argue that the instantaneous change of quantum states formalized, e.g., in terms of von Neumann's reduction postulate, could be an immanent[11] process occurring within what for the consciousness of an 'observer' is a 'specious' (i.e., non-vacuous) present which naturally corresponds to a temporal point in classical physics and generally in conventional science. Without intending to enter into the specifics of phenomenological analysis which, being a descriptive a priori philosophical theory, would carry us too much adrift I only point out that the specious present in Husserlian phenomenology represents a non-vacuous temporal 'point' in the sense of an original impression a priori linked with an immediate past and an a-thematic expectant future.[12] In this sense one may have, on the one hand, an exegetic context to interpret temporal points as non-individuals, i.e., as open sets in an appropriate topology (undertaken in the next section), and, on the other hand, certain clues on the extra-theoretical level into the superposition of states as in the case of von Neumann's reduction postulate. This means a new subjectively founded and yet non-reductionistic interpretation of the spatiotemporal 'there' in which the reduction postulate comes into effect, more concretely whether it does so somewhere in the broad environment of the system, or in the proximity of the measuring apparatus, or yet in the mind of the observer in terms of the triangle measured system - measuring apparatus - 'observer'.

Therefore temporal points in quantum outlook may have an inner content possibly

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10 For a phenomenologically based interpretation of quantum 'ontology' the interested reader may look at [27], [28] and [13].

11 This is a phenomenological term meaning what is inner, i.e. 'co-substantial' in non-physicalistic terms, to consciousness in contrast with what is external to it, e.g. the objects of real world experience. The interested reader may look at Husserl's *Ideas I* for a deeper knowledge of the fundamentals of phenomenology, [18].

12 See [19], pp. 30-33 and 54-57.

expressed and elaborated on the formal level in terms of topological openness, even as topological properties are mathematically meant as spatial properties, to the extent that they may be considered representative of an instant process of objectification-in-becoming, pointing therefore to an 'inner horizon' of the objective being as being constituted. The 'instantaneity' of the change of states and the application of von Neumann's projection postulate to capture in formal language what is in essence a process-in-becoming within any conceivable temporal unit may actually serve as prompts to the possibility of topologizing quantum temporal (e.g., history) processes. It must be pointed out that while in classical physics field quantities may be taken as properties of space-time points, in the case of quantum field theory field quantities may not be well-defined at space-time points, in view of the difficulties in establishing locationally exact quantum states, which means that quantum field properties may be 'spilled' over space-time regions, a fact that from a certain viewpoint reduces to the nature of space-time itself. This means that one has to address both the topological content of the notions of spatiotemporal point vs. spatiotemporal region and, more important, the fundamental role spatiotemporal topology acquires as a bearer of a system of relations between physical objects, an outstanding example being the relativist account of space-time in general relativity.

In this connection I point to Krause's claim that a "physical quantum mechanical space, whatever it means, is not Hausdorff"[13] ([26], p. 198). This is put in the sense that two entangled quantum systems cannot be set in complete isolation from each other which is also implied by the fact that there exist non-local correlations. Krause further remarks that "although quantum systems (*auth. note*: this term is used in his article interchangeably with that of quantum particles) are taken as punctual in the mathematical setting, from the physical point of view they cannot be taken as exactly punctual, that is, as precisely localized objects" (ibid., p. 199). Even in taking quantum systems as not represented by points but as 'something' confined in non-empty open balls in 'classical' Newtonian space-time, these balls thought apart at a distance greater than de Broglie's wavelength $\lambda = \frac{h}{p}$, p the momentum of the particle, the quantum systems gain nothing more than a mock individuality (as particles 'existing' in the surroundings of the first open ball, or of the second, etc.) which may be instantly lost upon interacting with others of a similar species. Obviously in such case there is a reasonable question on the legitimacy of the notion of identity in a quantum context. The fact that quantum particles may be detected as not punctually existing but as found 'somewhere' in some topologically open neighborhood should not prevent in principle to be named as such by an experimentalist during a measurement experiment, yet this is a mock individuality to the extent that it is not a self-standing one susceptible to determine

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[13] In general topology a topological space X is called Hausdorff if any two distinct points $x, y \in X$ can be separated by disjoint neighborhoods, i.e., $V_x \cap V_y = \emptyset$.

an individual identity. This claim may be further bolstered in case the implications of particle permutations are encoded in the topology of a space, e.g., by identifying certain points of such space, those corresponding to particle permutations, while at the same time adopting irreducible assumptions, for instance, some form of 'impenetrability assumption' to ensure, that no two particles (which are at least not bosons) occupy the same point of the reduced configuration space ([14], p. 9).

Assuming a co-constitutive role of time in quantum processes, one may reasonably associate the subjective sense ascribed to the temporal factor with the concept of quantum individuality, the latter concept still a hotly debated issue among theoretical physicists and epistemologists. Individuality, as associated with time, in the sense of a 'general something' regardless of material or any other 'thingness' content may be reduced to what as actual presence (e.g., in terms of a quantum measurement) is indecomposable as such, namely as the non-eliminable fulfilling of an intentional directedness in concrete actuality. In this virtue quantum individuality in the particular approach is inalienably associated with the subjective (inner) temporality as the sole ground of making present itself as the concretization of an each time intentional 'something' presupposing by means of evidence a time-constituting intentional consciousness. I do not intend to enter further into the intricacies of the phenomenological reduction that might be at play here as this could lead us far astray into an esoteric philosophical domain. The reader who wishes to delve more deeply into the Husserlian writings on these matters is referred to Husserl's *Ideas I* and the *Lessons on the Phenomenology of Inner Time-Consciousness*; [18] and [19].

Leaving deeper phenomenological inquiry aside, it is still important to cite a propos Krause's and Coelho's earlier claim in [25] that the mathematical structure of quantum mechanics should have a non-trivial rigid expansion (i.e., one not obtained by trivially adjoining the ordinal structure) whose physical intuition is that quantum objects are somehow intrinsically individuals. However this is contradicted later in [5] in which Arenhart and Krause espouse the view that the non-individuality of quantum particles, except for accounting for certain naturalistic concerns, should be also preferred in that it fits better with the claims of quantum theory and is better equipped on the formal level. Moreover this is a position running against Dorato and Morganti's thesis in [11], namely that one may find positive reasons to assume quantum individuality as an ungrounded fact, i.e., as a primitive 'essence' extraneous to the expressional means of a theory.

5. In what sense topology may be relevant with quantum history propositions

5.1. Preliminaries

Given that a major focus of this paper is the ways a topological conception of points vs open intervals (or sets) may influence concrete quantum situations in terms of proper theory, I undertake to address along these lines the question of single-time vs continuous-time quantum history propositions.

As it is known the role of time in quantum physics is unique in that it is not an 'internal' variable of the system and that all elementary quantum observations are 'instantaneous'. It is also known that in such case, i.e., in instantaneous measurements, one can extract from commutative C^*-algebras of bounded operators on a Hilbert space a corresponding Hausdorff topological space by the Gel'fand-Neymark theorem ([4], pp. 226-229). However as instantaneous measurements can only be an idealization and consequently measurements must be extended in time and space, there seems to be no fully articulated theory of measurements over time that maintain quantum coherence in the duration. A theory most likely to have a significant impact in these terms is the theory of quantum histories developed mainly by C. J. Isham, R. D. Sorkin, J. Hartle, R. Griffiths and others. This is especially true for the theory of consistent histories which are conceived so a to yield additive probabilities for any pair of different histories in a set of exclusive and exhaustive alternatives. Yet while histories theory has proved a convenient mathematical tool in describing a quantum situation over time in taking account of decoherence effects, e.g., in the calculation of systematic errors in an experiment, it still falls short of answering a key ontological question of quantum theory, namely of what actually occurs in a quantum measurement or of what is real if this term has a sound meaning. On this account, "At the end of a measurement, only one actual datum occurs. All those that were also possible are condemned. One does not explain the uniqueness of reality whereas reality is defined by philosophers through its uniqueness" [31], p. 282). As will be seen in the next my intention is to show how to take advantage of the histories discourse to provide a topological solution to a question relating to the continuity of lattice-preserving maps of history propositions for single moments of time.

Quantum causal histories are defined as histories that are both quantum mechanical and causal in the sense that observations made inside the system are closely related to causality in such a way that in a kind of 'quantum mechanical relativistic theory' an observer internal to the system splits the history of the system into a future, a past, and, in assuming a finite speed of propagation of information, an elsewhere. On the assumption that a discrete causal ordering (formalized in terms of a causal set) is a sufficient description of the fundamental past/future ordering of observations internal to a system, one may conceive of a quantum causal history as constructed by attaching finite-dimensional Hilbert spaces to the events of the causal set. We may normally consider tensor products of Hilbert spaces on events that are spacelike separated and define quantum histories by local unitary evolution maps between such sets of space-like separated events. The conditions of reflexivity, antisymmetry and transitivity that hold for a causal set are then naturally transformed by means of a quantum history into conditions on the evolution operators. We should keep in mind that transitivity was first imposed on causal sets because it is implied by the causal structure of Lorentzian spacetimes. It is notable that transitivity does not just encode properties of the ordering of events but also the fact that a Lorentzian manifold is a point set. It is known that in general relativity an event is a point and this has been integrated into the causal set approach ([29], p. 9).

Generally a history proposition corresponds to certain properties of a physical system at successive instants of time. To the extent that in quantum theory a property (or a proposition about it) is represented by a projection operator, a discrete-time history α will correspond to a sequence of projectors $\widehat{\alpha}_{t_1}, \widehat{\alpha}_{t_2}, .., \widehat{\alpha}_{t_n}$ each labeled by a corresponding time-point. C. J. Isham initiated in [20] and [21] a histories theory in which quantum logic is preserved by representing a history proposition as a projection operator on a tensor product of the Hilbert spaces of the canonical theory, $\mathcal{V} = \otimes_{t_i} H_{t_i}$. Then this history proposition will be written as $\widehat{\alpha} = \widehat{\alpha}_{t_1} \otimes \widehat{\alpha}_{t_2} \otimes ... \otimes \widehat{\alpha}_{t_n}$.

In standard quantum logic a history c is a finite set $(c_{t_1}, c_{t_2}, ..., c_{t_n})$ of single-time propositions. In a Hilbert space realisation it is a corresponding collection $(\widehat{c_{t_1}}, \widehat{c_{t_2}}, ..., \widehat{c_{t_n}})$ of projection operators that can be taken as an element of the direct sum $\oplus_{t \in \{t_1, t_2, ..t_n\}} B(\mathcal{H}_t)$ of n copies of the algebra $B(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$. It follows that $(\widehat{c_{t_1}}, \widehat{c_{t_2}}, ..., \widehat{c_{t_n}})$ is itself a projection operator which is what it is asked for a representative of a proposition to imply that the history c is 'realized' ([20], p. 18). To proceed from a Hilbert space representation of the lattice of single-time propositions to a full lattice of projection operators so as to incorporate arbitrary temporal supports (i.e., non-homogeneous histories), Isham has suggested the construction of infinite tensor products of copies of the algebra $B(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$ (ibid., p. 23).

Let us now consider a partition $I = \{t_1, ..., t_n\}$ of an interval T of the real line and construct through a pair of such discretizations I and $I^{'}$ of T the corresponding Hilbert spaces of the canonical theory H^I and $H^{I^{'}}$. We define the Hilbert space $\mathcal{V}$ as the continuous tensor product of Hilbert spaces along the time interval T, i.e., $\mathcal{V} = \otimes_{t \in T} H_t$.

Consider the map $J_{I,I'}$ from the cartesian product $H^I \times H^{I^{'}}$ of the Hilbert spaces above to the cartesian product $\mathcal{V} \times \mathcal{V}$, $J_{I,I'} : H^I \times H^{I^{'}} \to \mathcal{V} \times \mathcal{V}$. How can this map be made an injection map that is lattice-preserving? It must be continuous which seems not to be true for single-time propositions, that is, for single moments of time. It might be in weak topology but it is insufficient to define an order-preserving map. The point is that if indeed the injection map $J_{I,I'}$ exists and preserves the lattice structures then Kolmogoroff's continuity theorem goes through and the decoherence functional[14] on H^T exists as an

[14] A decoherence functional in terms of matrix formalization is defined to be a complex-valued function of a pair of histories $\mathrm{d}(a, b) = \mathrm{Tr}(\widehat{C}_a^{\dagger} \overline{\rho}_0 \widehat{C}_b)$, with $\overline{\rho}_0$ the density matrix describing the system at time $t = 0$. A

inductive limit of the decoherence functional defined on $H^I \times H^{I'}$ for all choices of I and I' ([3], p. 3233). If one accepts the restriction to an Abelian sublattice (e.g., corresponding to propositions about position) the map $J_{I,I'}$ need not be a continuous, linear map but simply a measurable map from the spectra of the corresponding operators $R^I \times R^{I'}$ to $R^T \times R^T$ which clearly exists, being virtually the same as in classical probability theory. However it is only on this restrictive condition that one can write the decoherence functional for continuous time as a limit of discrete-time ones, since one can have a continuous-time decoherence functional for each subalgebra but not one defined on the whole of $P(\mathcal{V})$, that is, on the lattice of projectors on the Hilbert space $\mathcal{V} = \otimes_{t\in T} H_t$ where T is the space of time instants taken as a subset of the real line.

In view of my intention it is instructing to consider the application of purely topological notions by Isham in [20], i.e., in the realization of the history propositions of standard quantum theory in the context of quasi-temporal theories, that is, theories which can arise from a 'non-conventional' conception of time evolution, e.g., theories in the context of quantum gravity and quantum field theory in a curved space-time. Based on the work of Hartle and Sorkin, respectively [16] and [36], Isham sought to show how their ideas can lead to an example of temporal supports for history propositions in a quasi-temporal situation.

The qualitatively new element in this 'space-time oriented' approach to quantum histories formalism is the application of topological properties to capture the difference between continuous-time and single-time history propositions.

In this theory an elementary history-filter is taken to be a collection of basic propositions $P(f_i, I_i),\ (I_i \subset R,\ i = 1, 2, .., n)$, in which the temporal support I_i of each test function f_i on the Lorentzian manifold $\mathcal{M}$ is compact, so that the propositions concerned are localized in space-time regions and causally connected in a certain way. Propositions involving regions that are disconnected can be generated from propositions localized in connected regions. This approach motivates the following definitions.[15]

- An open subset $A \subseteq \mathcal{M}$ is a basic region of $\mathcal{M}$ if A is connected and has a compact closure. **(1)**

- A temporal support is a collection $S = \{O_1, O_2, ..., O_n\}$ of basic regions O_i whose closures are non-intersecting such that for each pair $O_i, O_j \in S$ either $O_i \prec O_j$ or $O_j \prec O_i$ or O_i, O_j are space-like separated, the latter term meant in a relativistic sense.

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set of exclusive and exhaustive histories is called consistent whenever $d(a, b) = 0$ for any pair of histories a, b. See [3], p. 3226.

15 See [20], pp. 29-30.

The ordering $\prec$ is not a partial ordering and is defined for any two subsets A, B of $\mathcal{M}$ by:

$A \prec B$ if: $A \cap B = \emptyset$, $J^+(A) \cap B \neq \emptyset$, $J^+(B) \cap A = \emptyset$,

where the chronological future $J^+(A)$ of A is defined to be the set of points of the manifold $\mathcal{M}$ that can be reached from A by future-directed, non space-like curves. (**2**)

- A temporal support $S^{'} = \{O_1^{'}, O_2^{'}, ..., O_m^{'}\}$ is said to follow another $S = \{O_1, O_2, ..., O_n\}$, denoted by $S \lhd S^{'}$, if:

  (i) the closures of the basic regions in S and $S^{'}$ are pairwise disjoint and

  (ii) $\bigcup_{i=1}^{n} O_i \prec \bigcup_{j=1}^{m} O_j^{'}$

  If $S \lhd S^{'}$, a semi-group combination law is defined by:

$$S \circ S^{'} = \{O_1, O_2, ..., O_n, O_1^{'}, O_2^{'}, ..., O_m^{'}\} \quad (\mathbf{3})$$

- A history filter is a collection of propositions $P(f_1, I_1)$, $P(f_2, I_2)$, ...$P(f_n, I_n)$ in which the support of each test function $f_i,\ i = 1, 2, .., n$ is the closure of the element O_i of a temporal support $S = \{O_1, O_2, ..., O_n\}$. (**4**)

- A temporal support S is nuclear (i.e., it cannot be written in the form $S_1 \circ S_2$) if no $O_i \in S$ can precede or follow any other $O_j \in S$, so that all $O_i \in S$ are space-like separated.

A nuclear temporal support (a 'time point') will be a connected set with compact closure in virtue of containing a sole basic region. It is clear that any temporal support S can be written in the form $S = S_1 \circ S_2 \circ ... \circ S_n$ where $S_i,\ i = 1, 2, ...n$ are nuclear supports.

In standard quantum theory, any history filter $c = (c_{t_1}, c_{t_2}, ..., c_{t_n})$ with temporal support $\{t_1, t_2, ..., t_n\}$ can be written as the composition

$$c = c_{t_1} \circ c_{t_2} \circ .. \circ c_{t_n} \quad (\mathbf{5})$$

in which the single-time propositions $c_{t_i}, i = 1, 2, ..., n$ are thought of as history filters themselves with temporal supports the singleton sets $\{t_i\}$. By definition (**5**), the single-time propositions c_{t_i} are nuclear histories and each t_i is considered a nuclear support. In Isham's approach of a general history theory a nuclear support can be viewed as an analogue of a 'point of time' in standard Hamiltonian quantum theory as it admits of no further temporal-type subdivisions. Naturally, a nuclear history filter is an analogue of a single-time proposition. Due to intrinsic (topologically generated) properties the class of all temporal supports may be related by homomorphism to the properties of the space $\mathcal{U}$ of history filters.

In the next, I give a tentative topological workout of the construction of a continuous map between single-time and continuous-time quantum history propositions only to stumble on the non-Abelian property of the corresponding lattice. This will motivate in conclusion a brief discussion of the ways the topological concept of points relates with a quantum concept of time instants.

5.2. Construction of a topology in which temporal points could be open sets

Let's call by $\mathcal{S}$ the set of all possible temporal supports in the sense of subsection 5.1, more concretely an arbitrary temporal support is thought of in terms of an arbitrarily large sequence of points of the space, whereas a nuclear support s_i is considered a single point in the sense of a singleton set in the underlying topology. We may think of this construction in parallel terms with the construction of the Baire space $\mathcal{N}$, in which the points are sequences of natural numbers. The Scott topology on $\mathcal{S}$ can be taken, in accordance with its definition in [34] (pp. 646-647), as the collection of all open sets O such that:

1. $s_i \in O \rightarrow \uparrow s_i \subseteq O$ where s_i is a nuclear support, s any temporal support and $\uparrow s_i = \{s;\ s_i \lhd s\}$.[16]
2. $\bigsqcup_{\uparrow} s \in O \ \rightarrow s \cap O \neq \emptyset$. This last condition is easily seen to be fulfilled in case s is any collection of nuclear supports. In that case s is the directed join of its finite prefixes s_i under $\lhd$ (i.e., $s = \bigsqcup_{\uparrow} s_i$), with at least one of them having non-empty intersection with O .

As a base of the Scott topology on $\mathcal{S}$ we may take the class of the sets of the form $\uparrow s_i$, (s_i finite, $i \in I$, I a countably infinite index set) under $\lhd$ ordering. Clearly each open set O of the Scott topology can be seen by definition to be a union of basic open sets $\uparrow s_i$.

Now in the refined Scott topology on $\mathcal{S}$ except for the basic opens $\uparrow s_i$ I take as opens also the complements $\mathcal{S} \setminus \uparrow s_i$. In this case the nuclear supports s_i are proved to be open.

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[16] In general the $\lhd$ ordering is a prefix ordering that generalizes the intuitive concept of a tree by introducing the possibility of continuous progress and continuous branching. More concretely, we start with an initial finite segment of elements that can be expanded indefinitely by preserving the order relation. More formally:

A prefix ordering is a binary relation $\leq$ over a set P which is antisymmetric, transitive, reflexive, and downward total, i.e., for all a, b, and c in P, we have that:

$a \leq a$ (reflexivity); if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry); if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity); if $a \leq c$ and $b \leq c$ then $a \leq b$ or $b \leq a$ (downward totality). In accordance with the approach to quantum causal histories taken in [29], I take the condition $a \leq c$ and $b \leq c$, in case a and b do not cross to imply $a \leq b$.

5.1 Proposition. *In the refined Scott topology on $\mathcal{S}$ as defined above the nuclear supports s_i are open.*

Proof: Let's take a basic open set $\uparrow s_i$. Then in the refined topology its complement $(\uparrow s_i)^c$ is also open which means that the intersection $\uparrow s_i \bigcap (\uparrow s_i)^c$ is open. Since we have that $\uparrow s_i \bigcap (\uparrow s_i)^c = \{s_i\}$, this means that $\{s_i\}$ is open $\diamond$

Now consider the map $f : \mathcal{S}_1 \to \mathcal{S}$ where $\mathcal{S}_1$ is the class of all nuclear supports and $\mathcal{S}$ is the class of all possible temporal supports. Let also $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ of all arbitrary unions of temporal supports defined by $\triangleleft$ prefix ordering as above in terms of the nuclear supports $s_i \in \mathcal{S}_1$, i.e. $\mathcal{S}^* = \{\bigcup_{i\in I}\{\uparrow s_i\}\}$.

Consider a refined Scott topology on $\mathcal{S}$ based on the definition of the prefix ordering $\triangleleft$ and take as opens all the arbitrary unions of the basic open sets of the form $\uparrow s_i$ (i.e. those in the the class $\mathcal{S}^*$). Then we can build a continuous map $f : \mathcal{S}_1 \to \mathcal{S}$ defined by: $f(s_i) = \bigcup_{i\in I}\{\uparrow s_i\}$, where $\bigcup_{i\in I}\{\uparrow s_i\} \in \mathcal{S}^*$ and $s_i \in \mathcal{S}_1$.

5.2 Theorem. *The mapping $f : \mathcal{S}_1 \to \mathcal{S}$, defined by $f(s_i) = \bigcup_{i\in I}\{\uparrow s_i\}$, where $\bigcup_{i\in I}\{\uparrow s_i\} \in \mathcal{S}^* \subset \mathcal{S}$ is a continuous mapping in the refined Scott topology on $\mathcal{S}$.*

Proof: Let f be defined by $f(s_i) = \bigcup_{i\in I}\{\uparrow s_i\} \in \mathcal{S}^*$. To the extent that to each nuclear support s_i corresponds a uniquely defined $\bigcup_{i\in I}\{\uparrow s_i\} \in \mathcal{S}^*$ the map f is by construction one-to-one. Then clearly for any given open set $\bigcup_{i\in I}\{\uparrow s_i\} \in \mathcal{S}^*$ corresponds a uniquely defined $f^{-1}(\{\bigcup_{i\in I} \uparrow s_i\}) = \{s_i\}$. By Proposition 5.1 we have that $\{s_i\}$ is open in the refined Scott topology and therefore the mapping f is continuous $\diamond$

5.3 Theorem. *The mapping $f : \mathcal{S}_1 \to \mathcal{S}$ defined in Theorem 5.2 preserves order.*

Proof: Consider for any $s_i,\ s_j \in \mathcal{S}_1$ the $\triangleleft$ prefix ordering following from their natural order as natural numbers in the form of singletons, i.e., let $s_i \triangleleft s_j$. Then $\bigcup_{i\in I}\{\uparrow s_i\} \subseteq \bigcup_{j\in I}\{\uparrow s_j\}$ for the latter term includes also the unions on the indexes between i and j, $i < j$. By definition $f(s_i) = \bigcup_{i\in I}\{\uparrow s_i\}$ and $f(s_j) = \bigcup_{j\in I}\{\uparrow s_j\}$ implying that $f(s_i) \subseteq f(s_j)$ $\diamond$

This relatively simple formalism may serve as a prompt to show that we might possibly construct an order-preserving, continuous map between single-time and continuous-time history propositions.

We consider further a scheme of a quantum version $Q\mathcal{S}$ of the partially ordered set of temporal supports. In [29] an event q in a causal set[17] is considered a Planck-scale

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[17] A causal set C is a partially ordered set whose elements are interpreted as the events of a history. An acausal set is a set of events within a causal set C that are all causally unrelated to each other in terms of the partial ordering of C.

quantum 'event', which is a temporal point-event in view of the topological formalism, and it is associated with a Hilbert space $H(q)$ that stores its possible states. The space $H(q)$ should be finite-dimensional which is consistent with the requirement that causal sets are finite. This given we may consider the assignment of a corresponding Hilbert space H_{s_i} to each finite temporal support s_i, in which all $s_{ij} \in s_i, j \in N$ are causally unrelated to each other. This may actually prompt the generation of tensor products of Hilbert spaces in $Q\mathcal{S}$ for such members s_i in the class $\mathcal{S}$ of all temporal supports.

We must take into account that in case two acausal sets are related by a relation $\triangleright$, $s_i \triangleright s_j$,[18] there needs to be an evolution operator between the corresponding Hilbert spaces:

$$G_{ij} : H(s_i) \longrightarrow H(s_j)$$

The poset $\mathcal{G}$ of acausal sets s_i under $\triangleright$ is proved to be reflexive, transitive and antisymmetric (ibid., p. 4). If we choose to preserve these properties of the causal ordering as analogous conditions on the evolution operators of the quantum theory this would imply that the quantum causal history will be a functor from the poset $\mathcal{G}$ to the category of Hilbert spaces.

5.3. Could it be a continuous map $J_{I,I'}$ between single-time and continuous-time propositions?

If we properly associate the space of temporal supports $\mathcal{S}$ of subsection 5.2 to the Hilbert space H of the canonical quantum theory, under the inner product proposed in the Appendix (**A**), one may be able to construct the mapping $J_{I,I'} : H^I \times H^{I'} \rightarrow \mathcal{V} \times \mathcal{V}$ (see subsec. 5.1, par. 7) such that it could be continuous at a single moment of time (Appendix (**B**)).

As stated in the introduction this tentative result seems *prima facie* a way of topologically treating the question of constructing a continuous map between single time and continuous time history propositions.

Yet it may not apply to non-Abelian lattices of propositions proper to a quantum context due to the non-commutativity of the prefix ordering $\triangleleft$ in the space of all temporal supports $\mathcal{S}$ as constructed in subsection 5.2. Consequently it may globally fail to answer the question of continuously passing from single time to continuous time histories or answer, for that matter, the question of writing continuous time decoherence functionals as limits of discrete time ones. At the very least, though, it may be meant as giving a clue to the possibility of the treatment of temporal points and more generally of time in quantum systems in terms

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[18] The relation $\triangleright$ is defined by: $s_i \triangleright s_j$ if s_i is a complete past of s_j and s_j is a complete future of s_i. See for details [29], p. 4.

of topological openness and also as giving a clue to the epistemological relevance this topological tinkering might have in the particular context.

In view of my approach and taking into account that quantum mechanics may in a certain sense be regarded as a first-order approximation of topological theory it is interesting to note that Schlesinger has tentatively established in [35] a quantization of quantum mechanics parallel to the way Isham has quantized in [22] the lattice $\mathcal{T}(X)$ of topologies on a given set X.

More specifically for a given set X, let $C(X)$ be the sublattice of the lattice $\mathcal{T}(X)$ of the topologies generated by the singleton subsets of X, i.e., each element T of $C(X)$ is generated by a family $\{x_i\}_{i\in I},\ x_i \in X$, with the singleton sets x_i open with respect to the topology $\mathcal{T}$. Schlesinger has proved, on the supposition that the topologies of the sublattice $C(X)$ are generated by the open singletons $x_i,\ \ i \in I$, that there exists an isomorphism between the sublattice $C(X)$ and the lattice of subspaces of a pre-Hilbert space H[19] generated by a fixed algebraic base of H. Schlesinger has also dealt with the question of elaborating a first-order approximation to the full lattice of topologies $\mathcal{T}(X)$ by including quantum superpositions of the elements of $C(X)$ which may be derived by the isomorphism referred to above when going to the subspace lattice of Hilbert space $\overline{H}$ ([35], pp. 1442-1443).

Concluding the section: After all the lingering question is the proper way to represent temporal points in quantum processes in topological terms, to the extent that time enters in quantum mechanical situations as an ‘external’ parameter to the system, in which case one will have to settle for an irreducible incompatibility between point-like discreteness and topological openness (as point ‘superfluity’) that may fall short of describing entangled quantum states-of-affairs prior to measurement. We should keep in mind that in an essential way the ‘incompatibility’ between the mathematical points as syntactical individuals of a formal-axiomatical theory with the first-order logic they imply in contrast with real internals as subsets of the mathematical continuum with their implied second-order logic have spawned a host of undecidable mathematical statements about infinity the most famous of which is the well-known *Continuum Hypothesis*. By the same token such kind of ‘incompatibility’ may determine, as a matter of fact, the limits of topological methods to resolve quantum-theoretical questions like the one dealt with in the above.

Further as alluded in section 4, this may have to do, well beyond the intertheoretical level, with the subjectively founded ways by which one may have a conception of an abstract, ‘indecomposable’ individual independently of any physicalistic objectivist constraints in contrast with a collection of individuals as immanent unity ontologically superfluous to their sum total. This is deep and thorny question in a certain sense affecting

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[19] A pre-Hilbert space is a space incomplete with respect to the norm induced by the inner product.

also the discussion on the epistemological foundation of general relativity and ultimately running through the epistemological edifice in its entirety. One could claim, for instance, that the space-time continuum emerges from the basic discrete events. From a formal point of view, the manifold model of general relativity should be recovered from an adequate large limit number of a discrete set theory endowed with a partial order structure that preserves causal relations. In this sense time and spatial dimensions should emerge in the process of constituting objective reality as a well-defined one along with the topological and metric properties we attribute to space-time. More than simply a superficially conceived deficiency of current mathematics to accommodate the breach between the discrete and the continuum, between events on the quantum scale and our daily causal reality, this essentially ontological question is perhaps at the same time one of the great epistemological challenges of our time.

Obviously there is an open field ahead in ontological and epistemological terms to clarify the influence of topological notions in the conception we have of objects and processes of quantum theory insofar as there is still much to be worked out in the integration of topological methods into questions of hard quantum science.

6. Appendix

(**A**) Given any two members of the poset $\mathcal{S}$ of temporal supports we define the distance between them to be 2^{-n}, where n is the first position at which they differ. Formally the metric defined globally on $\mathcal{S}$ is:

$$d(s_i, s_j) = \text{inf}\{2^{-n};\ s_i \upharpoonright n = s_j \upharpoonright n\}$$

where $s_i \upharpoonright n$ denotes the truncation of s_i (i.e., the deletion of all terms of s_i after the first n). I note that the definition of the metric d is based on the agreement of the initial segments of s_i and s_j and that the infimum is preferable than the minimum to allow for the possibility that $s_i \upharpoonright n = s_j \upharpoonright n$ for all n. The topology induced is the one having as a base of open sets all sets of the form $\{s_i\}$ and $\uparrow s_i$ (see: [34], p. 703). This topology which is a refinement of Scott topology may be seen to coincide with the topology on $\mathcal{S}$ considered in subsec. 5.2.

It follows that $d(s_i, 0) = \text{inf}\{2^{-n};\ s_i \upharpoonright n = 0 \upharpoonright n\} = 2^{-n}$ and consequently the norm of s_i is defined as $\| s_i \| = d(s_i, 0) = 2^{-n}$. Naturally we may define an inner product $< s_i, s_j >= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} s_i s_j$ over the set of all temporal supports $\mathcal{S}$, provided that the series converges, and such that $< s_i, s_i >= (\| s_i \|)^2 = 2^{-2n} > 0$ for each finite n. This can be easily proved to satisfy the conditions of the definition of a Hilbert space

- $< a s_i + b s_j, s_k >= a < s_i, s_k > + b < s_j, s_k >$ and
- $\overline{< s_i, s_j >} =< s_j, s_i >$.

(**B**) Let T denote the set of all instants of time and topologize T by taking it to be the poset of all sequences of temporal supports of $\mathcal{S}$ under the subspace topology of Scott applied to the maximal elements of $\mathcal{S}$. The resulting space is a Baire space[20] known to be homeomorphic to the subspace of irrationals of the set of reals R under Euclidean topology. On this assumption we can in principle construct the tensor product of Hilbert spaces for the continuous time situation by appealing to continuous maps $\psi(.)$ from the space T to the Hilbert space H, that happen to be measurable in the way described by Anastopoulos in [3], (pp. 3231-3232). Standard quantum theory can be recovered by considering history propositions as corresponding to projection operators on the Hilbert tensor product space $\mathcal{V} = \otimes_{t \in T} H_t$, where H_t is a copy of the Hilbert space of the canonical theory indexed by t.

As stated before if we try to define an operator on $\mathcal{V} = \otimes_{t \in T} H_t$ corresponding to an observable at a sharp moment of time then, given that a point on the real axis is of measure zero, we run into problems by the implication of the delta function. Therefore the predominant view is that one cannot continuously embed the lattice of single-time propositions into the lattice of history propositions in the case of continuous time.

In spite of this difficulty, my approach to a possible construction of the map $J_{I,I'}$ as a continuous one based on the refined Scott topology on the class $\mathcal{S}$ of temporal supports serves the purpose of showing the influence topological constructs, in this case the specific topology of temporal supports, might have in bringing out new approaches to quantum histories theory as being time-associated. More than that, this is meant as an exercise on the ways topological notions, the one of openness in this case, may affect quantum theoretical questions on the level of theory as such independently of any peculiarities that might be registered on the observational level.

Ideally the map $J_{I,I'}$ would be a continuous mapping from $H^I \times H^{I'}$ to $\mathcal{V} \times \mathcal{V}$, with $I, I' \subseteq T$ and the time interval T topologized on the basis of the refined Scott topology on $\mathcal{S}$ (as homeomorphic to T). On this condition and under the product topology on the corresponding spaces the map $J_{I,I'}$ would be defined as follows:

Let $I = \{t_1, \ldots t_n\}$ and $I' = \{t'_1, \ldots t'_n\}$ be partitions of an interval T of the real line. Define $a_I : H^I \to \mathcal{V}$ and $a'_I : H^{I'} \to \mathcal{V}$ so that $J_{I,I'} : H^I \times H^{I'} \to \mathcal{V} \times \mathcal{V}$ can be expressed as $J_{I,I'} = a_I \times a_{I'}$. The map a_I should be of the form $a_I : (|\phi_1\rangle, |\phi_2\rangle, ..., |\phi_n\rangle) \to |\Phi\rangle$ for $(|\phi_1\rangle, |\phi_2\rangle, ..., |\phi_n\rangle) \in H^I$ and some $|\Phi\rangle \in \mathcal{V}$.

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[20] See [34], p. 647.

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