

Original Paper

A model of wave function collapse in a quantum measurement of spin as the Schrödinger equation solution of a system with a simple harmonic oscillator in a bath

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Abstract: We present a set of exact system solutions to a model we developed to study wave function collapse in the quantum spin measurement process. Specifically, we calculated the wave function evolution for a simple harmonic oscillator of spin $\frac{1}{2}$, with its magnetic moment in interaction with a magnetic field, coupled to an environment that is a bath of harmonic oscillators. The system's time evolution is described by the direct product of two independent Hilbert spaces: one that is defined by an effective Hamiltonian, which represents a damped simple harmonic oscillator with its potential well divided into two, based on the spin and the other that represents the effect of the bath, i.e., the Brownian motion. The initial states of this set of wave functions form an orthonormal basis, defined as the eigenstates of the system. If the system is initially in one of these states, the final result is predetermined, i.e., the measurement is deterministic. If the bath is initially in the ground state, and the wave function is initially a wave packet at the origin, it collapses into one of the two potential wells depending on the initial spin. If the initial spin is a vector in the Bloch sphere not parallel to the magnetic field, the final distribution among the two potential wells is given by the Born rule applied to the initial spin state with the well-known ground state width. Hence, the result is also predetermined. We discuss its implications to the Bell theorem [1]. We end with a summary of the implications for the understanding of the statistical interpretation of quantum mechanics.

Keywords: wave function collapse; Bell theorem; Schrödinger's cat problem

1. Introduction

The quantum measurement problem is related to the foundations of quantum mechanics. In particular, the collapse of the wave function during quantum measurement is crucial to the understanding of quantum mechanics. "The inability to observe such a collapse directly has given rise to different interpretations of quantum mechanics"[2]. We have chosen to use a model for the quantum measurement process to clarify the collapse of the wave function.

To study the collapse of the wave function, we are interested in the simplest example of a dissipative system. Such a system, i.e., a harmonic oscillator coupled to an environment, which is a bath of harmonic oscillators, has been the subject of extensive studies [3–19]. A special case of this system, the Ohmic case discussed by Caldeira and Leggett (to be defined later), is exactly solved using the path-integral with the density matrix diagonalized in [3–5], which indicates a wave function collapse. Following these works, we solved the wave function evolution directly in this special case as the solution of the Schrödinger equation[20], where even though we did not focus on the description of the wave function collapse, it is obvious in the solution.

In this paper, for the Ohmic case of the system, we shall focus on the wave function collapse and introduce spin $\frac{1}{2}$ into the simple harmonic oscillator, with its magnetic moment interacting with a magnetic field aligned to the axis of the oscillation in a direction z . The bath of harmonic oscillators serves as a model of a detector. The potential well of the simple harmonic oscillator is split into two potential wells according to whether the spin is in $+z$ or $-z$ direction. Thus, this models a spin measurement process.

We consider the problem the same as discussed by Caldeira and Leggett (CL) [3] and [20], except that in this case the main oscillator has spin: a harmonic-oscillator system (the dissipative system) with coordinate q on z axis, mass M , and frequency $(\omega_0^2 + \Delta\omega^2)^{\frac{1}{2}}$, interacting with a bath of N harmonic oscillators of coordinates x_j , mass m_j , and frequency ω_j , where $\Delta\omega^2$ is a shift induced by the coupling already discussed by CL. The Hamiltonian of the system and the bath is

$$H = \frac{p^2}{2M} + \frac{1}{2}M(\omega_0^2 + \Delta\omega^2)q^2 - MB\sigma_z q + q \sum_{j=1}^N c_j x_j + \sum_{j=1}^N \left(\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2 x_j^2 \right)$$

$$\sigma_z |+\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sigma_z |-\rangle = -|-\rangle = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \psi = \begin{pmatrix} \psi_+(q) \\ \psi_-(q) \end{pmatrix}$$

The interaction of the spin $\frac{1}{2}\sigma$ dipole moment with the magnetic field contributes to the term $-MB\sigma_z q$, where z is the projection of the Pauli matrix σ on z and ψ is the main oscillator's wave function. The simple harmonic oscillator's potential well is divided into two potential wells based on the spin.

Here, we outline the solution to this problem with examples and a possible interpretation:

In Section 1, despite the addition of the new term $-MB\sigma_z q$ to the previous Hamiltonian in [20], we show that this system can be solved almost exactly the same way. The system's time evolution is described by the direct product of two independent Hilbert spaces. One of them is defined by an effective Hamiltonian, which represents a damped simple harmonic oscillator with two split potential wells due to spin, and the other represents the effect of the bath, i.e., the Brownian motion.

We present a set of exact solutions of the system that all collapse to a δ -function centered at a specified position and time depending on whether the spin eigenstate is parallel or opposite to the magnetic field, with a small displacement determined by the eigenstates of each bath oscillators. The initial states of this set of wave functions form an orthonormal basis in the Hilbert space of the system, which is defined as the "eigenstates of the system." Hence, if the system is initially in one of these states, the final result is predetermined, i.e., the measurement is deterministic.

In Section 2, if the system is initially a superposition of these states, the final distribution is a superposition of δ -functions for a sufficiently long time after the damping time. Each has a probability amplitude determined by the corresponding eigenstate's probability amplitude in the initial state at time $t = 0$, with the center position determined by the spin eigenstate of the main oscillator and the contribution from each bath oscillator x_j . Once the distribution is determined from the initial state by the Born rule, the final distribution is also predetermined. As a result, we only need to apply quantum mechanics' statistical interpretation to the initial state; the final distribution is already determined by the Schrödinger equation.

One of the examples we shall discuss is the special case where the initial states of the bath oscillators are in their ground states at absolute temperature zero. In this case, the initial state of the bath is not an "eigenstate of the system," as we defined it above, but a superposition of the "eigenstates of the system," even though it is in the ground state (we remark here that the system is initially not in equilibrium even though the bath is in the ground state). The main harmonic oscillator is assumed to be initially a wave packet at the origin, i.e., in the middle between the two split potential wells. If the initial spin state is one of the two eigenstates of the main oscillator, for example, $|+\rangle$, the final state would be at the bottom of one of the two potential wells, that is, the one with $z > 0$, with a narrow spread due to the contributions from the bath oscillators which is the Brownian width. We find the width approximately agrees with the well-known width of the ground state for the simple harmonic oscillator if the damping time is much longer than its period.

In Section 3, we discuss another example where the initial spin state is a vector in the Bloch sphere not parallel to the magnetic field, i.e., it is the eigenstate of σ_θ where θ is the angle between the vector and the z direction. In this case, even though the initial state is

a pure state, it is not the eigenstate of the system; it is a superposition of two eigenstates of the system. This state is not an eigenstate of the spin measurement of σ_z because the magnetic field is not in the θ direction. The probability of the two eigenstates is given by the Born rule applied to the initial state, i.e., $\cos^2(\frac{\theta}{2})$ for $z > 0$, $\sin^2(\frac{\theta}{2})$ for $z < 0$, in spite of the initial state being a pure state. We shall demonstrate that the final distribution has two distinct peaks at the centers of the two potential wells accordingly, as shown in Fig.1. Again, the final result is determined by the Schrödinger equation according to the initial probability distribution.

If the bath temperature is not zero, the system is in a mixed state specified by a density matrix ρ according to the Boltzmann distribution. Still, we found the result is similarly determined by initial distribution.

In Section 4, we summarize the result and its implications in the Bell theorem. Because the discussion is only for a specific model, the interpretation is only suggestive. Applying the results of Sections 1, 2, and 3, we establish that the analysis of two separate spin measurements $\sigma_1 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ at remote positions A and B on two entangled particles in the Bell theorem is equivalent to the analysis of two measurements $-\sigma_2 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ at time $t = 0$ on particle 2 at the origin.

Because $\sigma_2 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ at time $t = 0$ do not form a set of commuting observables, if $\sigma_2 \cdot \vec{a}$ is definite, then $\sigma_2 \cdot \vec{b}$ is uncertain. Hence the probability distribution is a consequence of the uncertainty principle. So long as the uncertainty principle is valid, there is no hidden variable theory to have a definite value for $\sigma_2 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ simultaneously. Therefore, there is no need to resort to “local hidden variable theory” as J. Bell did to prove Bell inequality. As a result, there is no “nonlocality” involved in the violation of Bell inequality when the quantum mechanics prediction is confirmed. Because the result is predetermined by the initial condition, the final distance of the two entangled particles in the experiment in the Bell theorem does not affect the probability distribution as long as their spin correlation is not destroyed by the environment before detection. Thus, the experiment’s agreement with quantum mechanics does not provide information related to non-locality. This result suggests a viewpoint that may aid in understanding the interpretation of Bell’s theorem. This result also implies that the Born rule at the end of the measurement can be derived from the Schrödinger equation as long as the Born rule is applied to the interpretation of the initial state.

Finally, Section 5 gives a summary of the implications for the understanding of the statistical interpretation of quantum mechanics.

2. Solution of a damped simple harmonic oscillator with spin in a bath

The dynamic equation for operators in the Heisenberg representation leads to the following set of equations of motion:

$$M\ddot{q} = -M\omega_0^2q - M\Delta\omega^2q + MB\sigma_z - \sum_j c_j x_j \tag{1}$$

$$m_j \ddot{x}_j = -m_j \omega_j^2 x_j - c_j q \quad (j = 1, 2, \dots, N)$$

Now, applying the Laplace transform [5] (the bars are our notations for the Laplace transform, and s is the Laplace transform of time t), Eqs. (1) can be used to eliminate the bath variables x_j to obtain the equation for q ,

$$M(s^2\bar{q} - sq_0 - \dot{q}_0) = -M\omega_0^2\bar{q} - M\Delta\omega^2\bar{q} + MB\sigma_z \frac{1}{s} - \sum_j c_j \frac{sx_{j0} + \dot{x}_{j0}}{s^2 + \omega_j^2} + \sum_j \frac{c_j^2}{m_j(s^2 + \omega_j^2)}\bar{q} \tag{2}$$

where $q_0, \dot{q}_0, x_{j0}, \dot{x}_{j0}$ are the initial values of the respective operators in the Heisenberg representation. Assuming the number of bath oscillators is large enough so that we can replace the sum over j by integration over ω_j , the coefficient of the last term can then be separated into two terms:

$$\int_0^{\omega_{\text{cutoff}}} \frac{c_j^2}{m_j} \frac{\rho(\omega_j)}{\omega_j^2} d\omega_j - s^2 \int_0^{\omega_{\text{cutoff}}} \frac{c_j^2}{m_j} \frac{\rho(\omega_j)}{\omega_j^2} \frac{1}{(s^2 + \omega_j^2)} d\omega_j \tag{3}$$

where $\rho(\omega_j)$ is the bath oscillator density, with an upper-frequency limit ω_{cutoff} . Following an argument similar to the one pointed out by CL [3], the requirement that the system becomes a damped oscillator with frequency ω_0 and a damping rate η in the classical limit, known as the "Ohmic friction" condition, leads to the following constraint:

$$\rho(\omega_j) = \frac{2\eta M}{\pi} \frac{m_j \omega_j^2}{c_j^2} \tag{4}$$

If the frequency renormalization constant $\Delta\omega^2$ is chosen to satisfy

$$M\Delta\omega^2 = \int_0^{\omega_{\text{cutoff}}} \frac{c_j^2}{m_j} \frac{\rho(\omega_j)}{\omega_j^2} d\omega_j = \frac{2\eta M}{\pi} \omega_{\text{cutoff}}$$

Then by observing Eqs.(3,4), it can be shown that for sufficiently large $\omega_{\text{cutoff}} \gg \omega_0$, the first term of Eq.(3) represents a frequency shift $\frac{2\eta M}{\pi} \omega_{\text{cutoff}}$, while the second term of Eq.(3) leads to a damping term $-\eta M s \bar{q}$, with the damping constant η .

The frequency is shifted to ω_0 . Then Eq.(2) is simplified, and its inverse Laplace transform yields the quantum Langevin equation, which is valid at time $t > 0_+$:

$$\ddot{q}(t) + \eta\dot{q}(t) + \omega_0^2q(t) = f(t) \tag{5}$$

with a constant magnetic force $B\sigma_z$ and the Brownian motion driving force

$$f(t) = B\sigma_z - \sum_j \frac{c_j}{M} \left(x_{j0} \cos(\omega_j t) + \dot{x}_{j0} \frac{\sin(\omega_j t)}{\omega_j} \right) \quad (6)$$

During the derivation, in order to carry out the integral in Eq.(2), we used the requirement of the inverse Laplace transform that s must pass all the singular points from the right side of the complex plane, and hence $\text{Re}(s) > 0$. Equations (5) and (6) are the equations of a driven damped harmonic oscillator with an external force, the solution of which is well known as a linear combination of the initial values at $q_0 - \sigma_z d, \dot{q}_0, x_{j0}, \dot{x}_{j0}$ and a displacement $\sigma_z d$, where $d \equiv \frac{B}{\omega_0^2}$:

$$q(t) = a_1(t) (q_0 - \sigma_z d) + a_2(t) \dot{q}_0 + \sigma_z d + \sum_j (x_{j0} b_{j1}(t) + \dot{x}_{j0} b_{j2}(t)) \quad (7)$$

$$x_i(t) = \alpha_{i1}(t) (q_0 - \sigma_z d) + \alpha_{i0}(t) \sigma_z d + \alpha_{i2}(t) \dot{q}_0 + \sum_j (x_{j0} \beta_{ij1}(t) + \dot{x}_{j0} \beta_{ij2}(t))$$

with $a_1(t) = \frac{\mu e^{-\nu t} - \nu e^{-\mu t}}{\mu - \nu}$, $a_2(t) = -\frac{e^{-\mu t} - e^{-\nu t}}{\mu - \nu}$, $\mu \equiv \frac{\eta}{2} + \omega i$, $\nu \equiv \frac{\eta}{2} - \omega i$ and here $\omega \equiv \left(\omega_0^2 - \frac{\eta^2}{4} \right)^{1/2}$ is the frequency shifted by damping. And $a_1(0) = 1, a_2(0) = 0, \dot{a}_1(0) = 0, \dot{a}_2(0) = 1$, (see Appendix I for $a_1(t), a_2(t), b_{j1}(t), b_{j2}(t)$, etc). All formulas are correct whether ω is real or imaginary. We have redefined the initial time as $t = 0_+$ to avoid a minor detail of the initial-value problem. The explicit expressions for $\alpha_{i0}(t), \alpha_{i1}(t), \alpha_{i2}(t), \beta_{ij1}(t), \beta_{ij2}(t)$ are well known in basic physics.

We emphasize that the use of the Laplace transform instead of the Fourier transform allows us to express $q(t)$ and $x_j(t)$ explicitly in terms of the initial values, as in Eqs. (7). Equations (7) serve as the starting point of subsequent discussions. We will proceed to find the Green's function of the full system, and hence the solution of the wave function in the Schrödinger representation. The result tells us in what sense the damped oscillator is described by an effective Hamiltonian without the bath variables and gives it a specific form. It also shows that under this condition, the wave function can be factorized, and that the main factor relevant to the damped oscillator is a solution of the Schrödinger equation with an effective Hamiltonian.

Equations (7) are correct both in classical mechanics and in quantum mechanics in the Heisenberg representation. We notice that $q(t)$ and $x_j(t)$ are both linear superpositions of $q_0 - \sigma_z d, \dot{q}_0, x_{j0}, \dot{x}_{j0}, \sigma_z d$ with c-number coefficients. The commutation rules between $q(t), \dot{q}(t), x_j(t), \dot{x}_j(t)$ are $[q(t), \dot{q}(t)] = \frac{i\hbar}{M}$, $[x_j(t), \dot{x}_j(t)] = \frac{i\hbar}{m_j}$, and operators $q(t)$ and $\dot{q}(t)$ commute with $x_j(t), \dot{x}_j(t)$. One can prove these commutation rules in two ways: (1) by direct computation, using the fact that at $t = 0$, they are correct, and they all commute with operator σ_z and (2) by the general principle that $q(t), \dot{q}(t), x_j(t), \dot{x}_j(t)$ are related by a unitary transformation to $q_0, \dot{q}_0, \sigma_z, x_{j0}, \dot{x}_{j0}$.

Equations (7) show that the operators $q(t)$ and $x_j(t)$ can each be written as a sum of two terms:

$$q(t) = (Q(t) + \sigma_z d) + \sum_j \xi_j(t) \tag{8}$$

$$x_i(t) = \zeta_j(t) + \sum_j X_{ij}(t)$$

where

$$Q(t) = a_1(t)(q_0 - \sigma_z d) + a_2(t)\dot{q}_0 = a_1(t)Q_0 + a_2(t)\dot{Q}_0 \tag{9}$$

$$\xi_j(t) = x_{j0}b_{j1}(t) + \dot{x}_{j0}b_{j2}(t)$$

$(Q(t) + \sigma_z d)$ and $\zeta(t)$ are linear in q_0, \dot{q}_0 , and σ_z and independent of x_{j0}, \dot{x}_{j0} , and $\xi_j(t)$ and $X_{ij}(t)$ are linear in x_{j0}, \dot{x}_{j0} , and independent of q_0, \dot{q}_0 , and σ_z . Thus $Q(t), \sigma_z d$ and $\zeta(t)$ are operators in one Hilbert space S_Q , while $\xi_j(t)$ and $X_{ij}(t)$ are in an independent Hilbert space S_X , and the full Hilbert space is a direct product of $S_Q \otimes S_X$.

We shall first analyze the structure of S_Q . To explicitly show that we are discussing the S_Q space, we define $Q_0 \equiv q_0 - \sigma_z d, \dot{Q}_0 \equiv \dot{q}_0 = -\frac{i\hbar}{M} \frac{\partial}{\partial q_0}$. Because $\sigma_z d$ is a constant operator and it commutes with q_0 and \dot{q}_0 , we have $[Q_0, \dot{Q}_0] = [q_0, \dot{q}_0] = \frac{i\hbar}{M}$. Thus, we can write $-\frac{i\hbar}{M} \frac{\partial}{\partial Q_0} \equiv \dot{Q}_0 = \dot{q}_0$ as a well-defined operator in the space of q_0, σ_z , i.e., S_Q . The eigenfunction of $Q(t)$ with an eigenvalue denoted by Q_1 , in the Q_0 representation, is easily calculated to be (see Appendix II),

$$u_{Q_1}(Q_0, t) = \left(\frac{M\omega e^{\frac{\eta}{2}t}}{2\pi\hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left[-i \frac{M}{2\hbar a_2} (a_1 Q_0^2 - 2Q_1 Q_0 + \phi(Q_1, t)) \right] \tag{10}$$

with ϕ as an arbitrary phase, i.e., a real number. This eigenfunction is related to Green's function

$$G(Q_1, Q_0; t, 0) = \langle Q_1 | U(t) | Q_0 \rangle,$$

where we denote the evolution operator by $U(t)$. To see this, we use the relation between Schrödinger operator \mathbf{Q}_S and Heisenberg operator $Q(t) \equiv \mathbf{Q}_H(t) = U^{-1}(t)\mathbf{Q}_S U(t)$.

Let $|Q_1 \rangle$ be the eigenvector of \mathbf{Q}_S with eigenvalue Q_1 , i.e.,

$$\mathbf{Q}_S |Q_1 \rangle = Q_1 |Q_1 \rangle$$

we see that $U^{-1}|Q_1 \rangle$ is the eigenvector of $Q(t)$ of value Q_1

$$Q(t)U^{-1}|Q_1 \rangle = \mathbf{Q}_H U^{-1}|Q_1 \rangle = U^{-1}\mathbf{Q}_S U U^{-1}|Q_1 \rangle = U^{-1}\mathbf{Q}_S|Q_1 \rangle = Q_1 U^{-1}|Q_1 \rangle$$

So $U^{-1}|Q_1 \rangle$ is proportional to $u_{Q_1}(Q_0, t)$. If we choose $u_{Q_1}(Q_0, t)$ as an orthonormal basis, then U is unitary: $U^\dagger = U^{-1}$.

Then, we have $u_{Q_1}(Q_0, t) = \langle Q_0|U^{-1}|Q_1 \rangle = \langle Q_0|U^\dagger|Q_1 \rangle = \langle Q_1|U(t)|Q_0 \rangle^* = G^*(Q_1, Q_0; t, 0)$, i.e., $G(Q_1, Q_0; t, 0) = u_{Q_1}^*(Q_0, t)$. Thus we have

$$G(Q_1, Q_0; t, 0) = \left(\frac{M\omega e^{\frac{\eta}{2}t}}{2\pi\hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left[i \frac{M}{2\hbar a_2} (a_1 Q_0^2 - 2Q_1 Q_0 + \phi(Q_1, t)) \right] \quad (11)$$

Next, we shall determine the arbitrary phase $\phi(Q_1, t)$, which is the phase of the eigenvectors of $Q(t)$. Using Eq.(9), we find the commutation rule for Q and \dot{Q} (see Appendix I):

$$[Q, \dot{Q}] = (a_1(t)\dot{a}_2(t) - \dot{a}_1(t)a_2(t)) [Q_0, \dot{Q}_0] = e^{-\eta t} [Q_0, \dot{Q}_0] = e^{-\eta t} \frac{i\hbar}{M}$$

Thus, we define the canonical momentum $P(t)$ as

$$P = Me^{\eta t} \dot{Q} = Me^{\eta t} (\dot{a}_1(t)Q_0 + \dot{a}_2(t)\dot{Q}_0) = Me^{\eta t} \dot{a}_1(t)Q_0 - i\hbar e^{\eta t} \dot{a}_2(t) \frac{\partial}{\partial Q_0} \quad (12)$$

and get the commutation rule $[Q(t), P(t)] = i\hbar$. The eigenfunction of $P(t)$ can be calculated in two ways: (1) we can calculate the eigenvector of $P(t)$ in the Q_0 representation using Eq.(12) and then use Green's function Eq.(11) to transform it into the $Q(t)$ representation and (2) the commutation rule $[Q(t), P(t)] = i\hbar$ requires that $P(t) = -i\hbar \frac{\partial}{\partial Q}$, so the eigenfunction of $P(t)$ with eigenvalue P_1 is $\exp[i \frac{P_1}{\hbar} Q]$. By comparing these two solutions, the arbitrary phase $\phi(Q_1, t)$ in Green's function is determined to be within a phase $\phi(t)$, which is independent of Q_1 . $\phi(t)$ is an arbitrary real function of time, except that $\phi(0) = 0$ so that it satisfies the condition that at $t = 0$, the Green's function becomes $\delta(Q_1 - Q_0)$. Thus, we obtain Green's function in the S_Q space:

$$G(Q_1, Q_0; t, 0) = \left(\frac{M\omega e^{\frac{\eta}{2}t}}{2\pi i\hbar \sin(\omega t)} \right)^{\frac{1}{2}} \exp \left[i \frac{M}{2\hbar a_2} (a_1 Q_0^2 - 2Q_1 Q_0 + e^{\eta t} \dot{a}_2 Q_1^2) - \frac{i}{\hbar} \phi(t) \right] \quad (13)$$

It is then straightforward to derive the Hamiltonian H_Q using the following relation:

$$H_Q = i\hbar \left(\frac{\partial}{\partial t} U_Q \right) U_Q^{-1}$$

and remember that the matrix elements of U_Q and U_Q^{-1} are Green's function and its conjugate. The result is

$$H_Q = e^{-\eta t} \frac{P^2}{2M} + \frac{1}{2} M e^{\eta t} \omega_0^2 Q^2 + \dot{\phi}(t) \tag{14}$$

Since ϕ is arbitrary except that $\phi(0) = 0$, we can take $\phi(t) = 0$. Therefore, we have derived the well-known effective Hamiltonian for the dissipative system. We emphasize that the expression for H_Q is derived here, whereas it is usually introduced by heuristic arguments.

Next, we shall analyze the effect of the bath. Similar to Eq.(9), we define the contribution of the bath oscillator j to the Brownian motion of the main oscillator as

$$\xi_j(t) \equiv x_{j0} b_{j1}(t) + \dot{x}_{j0} b_{j2}(t)$$

where $b_{j1}(t), b_{j2}(t), \dot{b}_{j1}(t), \dot{b}_{j2}(t)$ all equal to zero at $t = 0$ (see Appendix I), so $\xi_j(t = 0) = 0$ and $\dot{\xi}_j(t = 0) = 0$, and hence in Eq.(8), $q(0) = q_0, \dot{q}(0) = \dot{q}_0$.

Similar to Eq.(10) we obtain the eigenfunctions $\theta_{\xi_{j1}}(x_{j0}, t)$ for ξ_j and $t > 0$. Using Dirac's notation we have

$$\begin{aligned} Q|u_{Q_1}, s\rangle &= Q_1|u_{Q_1}, s\rangle \\ \xi_j|\theta_{\xi_{j1}}\rangle &= \xi_{j1}|\theta_{\xi_{j1}}\rangle, \end{aligned}$$

where $s = \pm 1$ is the eigenvalue of σ_z :

$$\sigma_z|s\rangle = s|s\rangle,$$

and

$$|u_Q, s\rangle \equiv u_Q(Q_0, t)|s\rangle.$$

Thus $|u_Q, s\rangle \prod_j \otimes |\theta_{\xi_j}\rangle$ is an eigenvector of $q(t)$, with eigenvalue of $q = Q + sd + \sum_j \xi_j$. Hence $Q = q - sd - \sum_j \xi_j$. In other words, the eigenvector of $q(t)$ with eigenvalue q is

$$|q, s, \{\xi_j\}\rangle = |u_{q-sd-\sum_j \xi_j}, s\rangle \prod_j \otimes |\theta_{\xi_{j1}}\rangle \tag{15}$$

The set of Hermitian $q, s, \{\xi_j\}$ is a set of commuting observables distinct from the set of $q, s, \{x_j\}$. The set of wave functions $|q, s, \{\xi_j\}\rangle$ forms a Hilbert space's orthonormal basis.

We first study the evolution of the damped simple harmonic oscillator wave function. Initially ($t = 0$) it is $u_{Q_1}(Q_0, t)$. Then, at time t , according to the discussion following

Eq.(10) , $G(Q_1, Q_0; t, 0) \equiv \langle Q_1 | U_Q(t) | Q_0 \rangle$, and $u_{Q_1}(Q_0, t) = \langle Q_0 | U_Q^{-1}(t) | Q_1 \rangle$, it evolves into

$$\begin{aligned} \int dQ_0 G(Q_2, Q_0; t, 0) u_{Q_1}(Q_0, t) &= \int dQ_0 \langle Q_2 | U_Q(t) | Q_0 \rangle \langle Q_0 | U_Q^{-1}(t) | Q_1 \rangle \\ &= \langle Q_2 | U_Q(t) U_Q^{-1}(t) | Q_1 \rangle = \langle Q_2 | Q_1 \rangle = \delta(Q_2 - Q_1) \end{aligned}$$

Similarly, we can confirm that if the wave function of the bath oscillator j is $\theta_{\xi_{j1}}(x_{j0}, t)$, at time $t > 0$, it evolves into $\delta(\xi_j - \xi_{j1})$ at time $t > 0$.

We emphasize that if the initial state (at $t = 0_+$) is $|q_1, s_1, \{\xi_{j1}\}\rangle$, then the wave function will evolve at time t into $\delta(q - q_1) \delta_{s, s_1} \prod_j \delta(\xi_j - \xi_{j1})$. At time $t = 0_+$, because $Q(t), \sigma_z$ and $\{\xi_j(t)\}$ are linear combinations of operators $q_0, \dot{q}_0, \sigma_z, x_{j0}, \dot{x}_{j0}$, they form a complete set of commuting observables of time $t = 0_+$. In other words, at time $t = 0_+$, we consider the unitary transform from q_0, σ_z, x_{j0} , to $Q(t), \sigma_z$ and $\{\xi_j(t)\}$ as a variable transform. The set of wave functions $|q_1, s_1, \{\xi_{j1}\}\rangle$ forms an orthonormal basis in the system's Hilbert space, which we call the "eigenstates of the system". As a result, if the system is initially (that is, at $t = 0_+$) in one of these states, $|q_1, s_1, \{\xi_{j1}\}\rangle$, the final result at time t is predetermined, i.e., the measurement is deterministic, with a definite value of $q_1, s_1, \{\xi_{j1}\}$.

3. The initial superposition of eigenstates determines the final probability distribution

If the wave function is initially $|\Psi_0\rangle = |\psi_0, s\rangle \prod_j |\chi_{j0}\rangle$, then to calculate the wave function at time t , we must expand Ψ_0 in terms of the eigenvectors $|q, s, \{\xi_j\}$ in Eq.(15), i.e., we must calculate

$$\psi(Q, s, t) = \langle u_Q, s, t | \psi_0, s \rangle = \int u_Q^*(q_0, t) \psi_0(q_0) dq_0 \langle s | s \rangle = \int u_Q^*(q_0, t) \psi_0(q_0) dq_0 \tag{16}$$

$$\chi_j(\xi_j, t) = \langle \theta_{\xi_j}, t | \chi_{j0} \rangle = \int \theta_{\xi_j}^*(x_{j0}, t) \chi_{j0}(x_{j0}) dx_{j0}$$

Notice that even though we label these functions by the parameter t , they are the wave functions at time $t = 0$. We label them with t only to show that we define them as the initial state that will evolve to specified $Q, \{\xi_j\}$ at time t . The probability amplitude for the system in state $|q_1, s_1, \{\xi_{j1}\}\rangle$ at time $t = 0_+$ is then

$$\begin{aligned} \Psi(q_1, s_1, \{\xi_{j1}\}, t) &= \langle q_1, s_1, \{\xi_{j1}\} | \Psi_0 \rangle = \langle u_{q_1 - s_1 d - \sum_j \xi_{j1}, s_1, t} | \psi_0 \rangle \langle s_1 | s_1 \rangle \prod_j \langle \theta_{\xi_{j1}}, t | \chi_{j0} \rangle \\ &\equiv \psi(q_1 - s_1 d - \sum_j \xi_{j1}, s_1, t) \prod_j \chi_j(\xi_{j1}, t) \end{aligned} \tag{17}$$

Because $|q_1, s_1, \{\xi_{j1}\}\rangle$ evolves into $\delta(q - q_1)\delta_{s,s_1} \prod_j \delta(\xi_j - \xi_{j1})$ at time t , the wave function at time t is

$$\int dq_1 \prod_j \int d\xi_{j1} \sum_{s_1} \delta(q - q_1)\delta_{s,s_1} \prod_j \delta(\xi_j - \xi_{j1}) \Psi(q_1, s_1, \{\xi_{j1}\}, t) = \Psi(q, s, \{\xi_j\}, t) \tag{18}$$

This equation seems to be redundant because it simply replaces $q_1, s_1, \{\xi_{j1}\}$ by $q, s, \{\xi_j\}$, so in the following, we shall only use Eq.(17) to get the wave function. However, the purpose of this redundant step is to point out that the final wave function is obtained by projecting the initial state onto the system eigenstates, which themselves are taken as the initial states. We emphasize here again that this probability amplitude is completely determined by the initial wave function $|\Psi_0\rangle$. More examples will be provided to demonstrate the implications of this point.

Notice that $\psi(Q, t)$ of Eq.(16) is the wave function in the Schrodinger representation with the effective Hamiltonian Eq.(14). Thus, we have connected the effective Hamiltonian approach to the dissipative system problem with the other approaches that take both the system and the bath into account. We also notice that, despite the fact that our $\Psi(q, s, \{\xi_j\}, t)$ is in a different representation than $\Psi(q, s, \{x_j\}, t)$, the usual probability interpretation remains valid: $\int \int \dots \int |\Psi(q, s, \{\xi_j\}, t)|^2 \prod_j d\xi_j$ is the probability density of finding the particle at q . Since this solution is very simple, it provides a simple way to analyze other complicated problems, e.g., studying the influence of Brownian motion on interference, on which we shall not elaborate..

Under certain conditions, for example, at low temperatures and when the system q is in highly excited states, the range of q is large enough that we can approximately write $\Psi(q, s, \{\xi_j\}, t) = \psi(q - sd, s, t) \prod_j \chi_j(\xi_j, t)$ for all the values of j that do not have a vanishingly small probability, $q \gg |\sum_j \xi_j|$. That is, the wave function is factorized, the dissipative system q can be described by the wave function $\psi(q - sd, s, t)$ only, and the Brownian motion can be ignored. As a result, it is worthwhile to investigate the width of the argument of the wave function due to Brownian motion, i.e., the mean value of $(\sum_j \xi_j)^2$ at time t . It can be calculated using the expression $\psi(Q, s, t)$ and $\chi_j(\xi_j, t)$ in Eq.(16), as shown in the example below.

As one example of the superposition of the ‘‘eigenstates of the system,’’ we assume the absolute temperature is zero; the initial state of the bath oscillators is in the ground state; the initial state of the damped simple harmonic oscillator is a wave packet at the origin ($q = 0$) with width same as the width of ground state of the simple harmonic oscillator $\sigma_q^2 = \frac{\hbar}{2M\omega_0}$ and the spin state is $|s \rangle = | + 1 \rangle$. The initial state of the damped harmonic oscillator is

$$\psi_0(Q_0) = \left(\frac{1}{2\pi\sigma_q^2}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{4\sigma_q^2}(Q_0 + d)^2\right)$$

where $Q_0 = q_0 - d$ and $d \equiv \frac{B}{\omega_0}$. Following the step of Eq.(17) by Gaussian integration, we find the wave function at time t as a Gaussian,

$$\begin{aligned} \psi(Q, t) &= \phi_0(Q, t) \exp\left(\frac{M\omega_0 - 2Qd - a_1d^2}{\hbar} \frac{1}{2(a_1 + i\omega_0a_2)}\right) \\ \phi_0(Q, t) &= \left(\frac{M\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \left(\frac{1}{a_1(t) + i\omega_0a_2(t)}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega_0}{2\hbar} \frac{\omega_0a_2(t) - i\dot{a}_2(t)}{\omega_0a_2(t) - ia_1(t)} e^{\eta t} Q^2\right) \end{aligned}$$

where $\phi_0(Q, t)$ is the solution when the magnetic field is turned off ($B = 0$). The functions $a_1(t), a_2(t), \dot{a}_1(t), \dot{a}_2(t)$, are given in Appendix I. The initial function $\psi_0(Q_0) = \phi_0(Q_0 + d, t = 0)$ is the function of the ground state of the oscillator with the centroid displaced to $-d$ but Q_0 is the position relative to d , so the centroid is at $q_0 = 0$.

The probability distribution of Q ignoring the contribution from the bath, is given by

$$\begin{aligned} |\psi(Q, t)|^2 &= \left(\frac{1}{2\pi\sigma_Q^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(Q + a_1d)^2}{2\sigma_Q^2}\right) \tag{19} \\ \sigma_Q^2 &\equiv \frac{\hbar}{2M\omega_0} (\omega_0^2a_2^2 + a_1^2) = \sigma_q^2 \left(1 + \frac{\eta^2}{4\omega^2} + \frac{1}{2} \frac{\eta}{\omega} \sin(2\omega t) - \frac{\eta^2}{4\omega^2} \cos(2\omega t)\right) e^{-\eta t} \end{aligned}$$

This is a wave packet that begins at $Q = -d$ (i.e., $q = 0$), oscillates around $Q = 0$ (i.e. $q = d$), and has its amplitude dampened to zero because $a_1(t) = e^{-\frac{\eta}{2}t} (\cos(\omega t) + \frac{\eta}{2\omega} \sin(\omega t))$. With damping time $2/\eta$, the width $\sigma_Q(t)$ also damps to zero, so the wave function collapses to the point at d . If the initial spin state is $s = -1$, it collapses to $-d$.

To include the Brownian motion from the bath, we have the bath oscillator initial wave function in ground state

$$\chi_{j0}(x_{j0}) = \left(\frac{m_j\omega_j}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m_j\omega_j}{2\hbar} x_{j0}^2\right)$$

The eigenfunction of $\xi_j|\theta_{\xi_{j1}}\rangle = \xi_{j1}|\theta_{\xi_{j1}}\rangle$ gives $\theta_{\xi_{j1}}(x_{j0}, t) = \left(\frac{m_j}{2\pi\hbar b_{j2}(t)}\right)^{\frac{1}{2}} \exp\left[-i\frac{m_j}{2\hbar b_{j2}(t)} (b_{j1}(t)x_{j0}^2 - 2\xi_{j1}x_{j0})\right]$. The derivation of $\theta_{\xi_{j1}}(x_{j0}, t)$ is very similar to the derivation of the eigenfunction u_Q of $Q(t)$ in Appendix II, (see Appendix I for the expression for $b_{j1}(t), b_{j2}(t)$). The projection of the ground state onto the eigenvector in Eq.(17) is found by another Gaussian integral as

$$\chi_j(\xi_j, t) = \langle \theta_{\xi_j}, t | \chi_{j0} \rangle = \left(\frac{m_j \omega_j}{\pi \hbar} \right)^{\frac{1}{4}} \left(\frac{1}{-ib_{j1} + \omega_j b_{j2}(t)} \right)^{\frac{1}{2}} \exp \left(-\frac{m_j}{2\hbar b_{j2}} \frac{\xi_{j1}^2}{(-ib_{j1} + \omega_j b_{j2}(t))} \right) \quad (20)$$

As a result, the contribution to the Brownian motion width derived from $\chi_j(\xi_j, t)$ is $\langle \xi_j^2 \rangle = \sigma_{\xi_j}^2 = \frac{1}{2} (b_{j1}^2 + \omega_j^2 b_{j2}^2) \frac{\hbar}{m_j \omega_j}$.

According to the discussion that follows Eq.(17,18), the width of the wave function

$$\Psi(q_1, s_1, \{\xi_{j1}\}, t) = \psi(q - sd - \sum_j \xi_j, s, t) |s\rangle \prod_j \chi_j(\xi_j, t)$$

can be calculated from the widths of $\psi(Q, t)$ in Eq.(19) and $\chi_j(\xi_j, t)$ in Eq.(20) as $\sigma_Q^2 + \langle (\sum_j \xi_j)^2 \rangle$ (see Appendix III for the derivation). Long after the damping time $2/\eta$, the width $\sigma_Q^2 \rightarrow 0$, and thus the Brownian width is given by

$$\langle \left(\sum_j \xi_j(t) \right)^2 \rangle = \int \int \dots \int \left(\sum_j \xi_j(t) \right)^2 \prod_j |\chi_j(\xi_j, t)|^2 d\xi_j = \sum_j \langle \xi_j^2 \rangle = \sum_j \sigma_{\xi_j}^2$$

From the bath's spectral density Eq.(4), and the expressions for $b_{j1}, b_{j2}, \omega, \omega_0, \eta$ in Appendix I, and Eq.(7), we find

$$\begin{aligned} \sigma_{\xi}^2 &\equiv \sum_j \sigma_{\xi_j}^2 = \sum_j \frac{\hbar}{2m_j \omega_j} (b_{j1}^2 + \omega_j^2 b_{j2}^2) = \int d\omega_j \rho(\omega_j) \frac{\hbar}{2m_j \omega_j} (b_{j1}^2 + \omega_j^2 b_{j2}^2) \quad (21) \\ &= \frac{\eta \hbar}{\pi M} \int d\omega_j \omega_j \frac{M^2}{c_j^2} (b_{j1}^2 + \omega_j^2 b_{j2}^2) \end{aligned}$$

As $t \rightarrow \infty$, this approaches

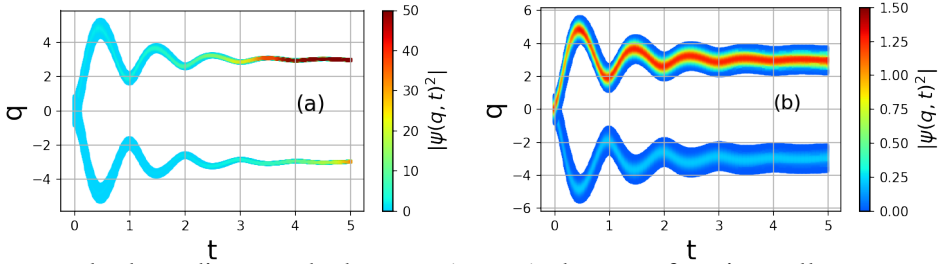
$$\sigma_{\xi}^2 = \sum_j \sigma_{\xi_j}^2 = \frac{\eta \hbar}{2\pi M} \int_0^\infty d\omega_j^2 \frac{1}{(\omega_0^2 - \omega_j^2)^2 + \eta^2 \omega_j^2} = \frac{\hbar}{2\pi \omega M} \left(\frac{\pi}{2} + \arctan\left(\frac{\omega^2 - \frac{\eta^2}{4}}{\eta \omega}\right) \right)$$

If $\eta \ll \omega$, i.e., if the damping time is much longer than the main oscillator period.

$$\langle \left(\sum_j \xi_j(t) \right)^2 \rangle \approx \frac{\hbar}{2\pi \omega M} \left(\frac{\pi}{2} + \arctan\left(\frac{\omega}{\eta}\right) \right) \approx \frac{\hbar}{2\omega M}$$

This Brownian width approximately equals the ground state of the simple harmonic oscillator.

Figure 1. (a) Wave function probability density without Brownian motion (b) with Brownian motion. The initial spin state is specified by a vector \vec{a} on the Bloch sphere with $\theta = \pi/4$ for this case. The probability for the spin up and down is both non-zero. The color scale is the probability density. The two wave packets of the spin up and down are well separated near the beginning. When Brownian motion is ignored, the wave function profile width approaches zero at $t \geq 5$. The wave function collapses to either at $q = d = 3$, or $q = -d = -3$ for $t > 5$, with the probability of being up and down as $\cos^2(\theta/2) = 0.8535$, and $\sin^2(\theta/2) = 0.1464$.



As a result, depending on whether $s = 1$ or -1 , the wave function collapses to one of the two split potential wells at $\pm d$ with a spread $\sigma_\xi = \sqrt{\frac{\hbar}{2\omega M}}$. As long as $d \gg \sqrt{\frac{\hbar}{2\omega M}}$, the spin measurement has a definite answer. From the discussion following Eq.(17,18), this result is completely predetermined by the initial spin state.

4. An example of the case when the Bloch vector of the pure state spinor is not aligned with the z axis

When the spin is in a pure state, the initial state can always be specified by a vector $\vec{a} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ on the Bloch sphere[21] as

$$|\vec{a}\rangle = \cos(\theta/2) |+\rangle + e^{i\phi} \sin(\theta/2) |-\rangle$$

Then the initial state $\Psi_0 = |\psi_0\rangle |\vec{a}\rangle \prod_j |\chi_{j0}\rangle$.

\vec{a} is not parallel to the z axis unless $\theta = 0$, or π . Its projection onto the initial system eigenstate $|q_1, s_1, \{\xi_{j1}\}\rangle$ in Eq.(15), given by Eq.(17,20) is

$$\begin{aligned} \Psi(q_1, s_1, \{\xi_{j1}\}, t) &= \langle q_1, s_1, \{\xi_{j1}\} | \Psi_0 \rangle = \langle u_{q_1 - s_1 d - \sum_j \xi_{j1}}, s_1, t | \psi_0 \rangle \langle s_1 | \vec{a} \rangle \prod_j \langle \theta_{\xi_{j1}}, t | \chi_{j0} \rangle \\ &= \psi(q_1 - s_1 d - \sum_j \xi_{j1}, s_1, t) \langle s_1 | \vec{a} \rangle \prod_j \chi_j(\xi_{j1}, t) \end{aligned}$$

According to the discussion following Eq.(17,18), at time t , Ψ_0 evolves into the wave function

$$\Psi_+(q, t) \equiv \Psi(q, 1, \{\xi_j\}, t) = \psi(q - d - \sum_j \xi_j, 1, t) \cos(\theta/2) \prod_j \chi_j(\xi_j, t) \quad (22)$$

$$\Psi_-(q, t) \equiv \Psi(q, -1, \{\xi_j\}, t) = \psi(q + d - \sum_j \xi_j, -1, t) e^{i\phi} \sin(\theta/2) \prod_j \chi_j(\xi_j, t)$$

The wave packet split into two, with $s = 1$ collapsing to $q = d$ with a probability of $\cos^2(\theta/2)$, and $s = -1$ collapsing to $q = -d$ with a probability of $\sin^2(\theta/2)$, both with a width $\sigma_\xi = \sqrt{\frac{\hbar}{2\omega M}}$, given by Eq.(21)

At temperature T the contribution from the Brownian width of the bath to the width of q is calculated by the bath density matrix as

$$\left\langle \left(\sum_j \xi_j(t) \right)^2 \right\rangle = \sum_j \frac{\hbar}{2m_j \omega_j} (b_{j1}^2 + \omega_j^2 b_{j2}^2) \coth\left(\frac{\hbar \omega_j}{2kT}\right)$$

Figure 2. (a) the width $\sigma_{Q\xi}(t)$, $\sigma_\xi(t)$, and $\sigma_Q(t)$; (b)Probability in the wave packet with spin up (magenta), down(cyan), and total (blue);

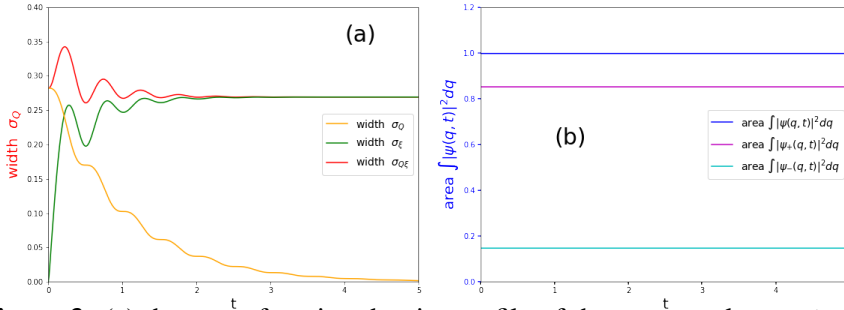
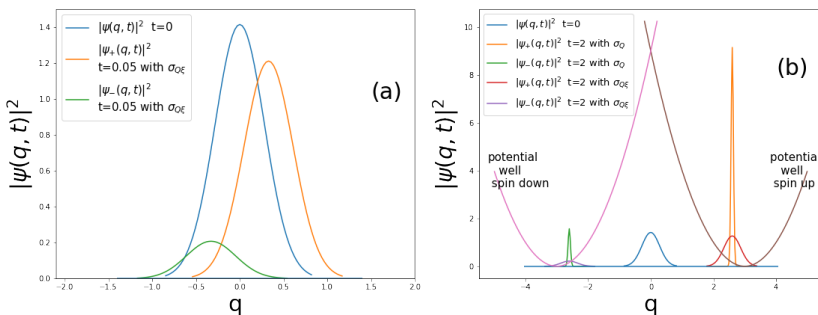


Figure 3. (a) the wave function density profile of the wave packets at $t = 0$ and very close to the beginning at $t = 0.05$. The two wave packets is already starting to separate. (b) Wave function density profile at $t = 0$ and $t = 2$ compares with and without Brownian motion included. We also indicate the positions of the two potential wells corresponding to spin up or down, with arbitrary scale. The two wavepackets are close but do not reach the bottom of the potential wells yet at $t = 2$.



This approaches the width in Eq.(21) as T approaches zero. The width increases with temperature.

When we apply Eq.(26) and Eq.(19) for $|\psi(Q, t)|^2$ given in Appendix III, we have

$$|\Psi_+(q, t)|^2 = \cos^2(\theta/2) \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{Q\xi}} \exp\left(-\frac{(q-d+a_1d)^2}{2\sigma_{Q\xi}^2}\right)$$

$$|\Psi_-(q, t)|^2 = \sin^2(\theta/2) \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{Q\xi}} \exp\left(-\frac{(q+d-a_1d)^2}{2\sigma_{Q\xi}^2}\right)$$

where the width of the wave function density is $\sigma_{Q\xi}^2 = \sigma_Q^2 + \sigma_\xi^2$.

For this abstract model, for a case with $\hbar = 1, M = 1$, and the main oscillator period $T_0 = 1, \omega_0 = 2\pi/T_0, \eta = 2$, we choose $d = 3$, at temperature $T = 0$. The angle between the spin direction and the z axis is chosen to be $\theta = \pi/4$. We plot the $|\Psi_+(q, t)|^2, |\Psi_-(q, t)|^2$ versus t in Fig.1a when the Brownian motion is ignored, i.e. with width σ_Q , and in Fig.1b with Brownian motion included, i.e., with width $\sigma_{Q\xi}$. The separation of the two wave packets is visible from the very beginning near $t = 0$.

The width in Eq.(19) $\sigma_Q^2(t) \rightarrow 0$ as $t \rightarrow \infty$. As a result, the wave packet is already approaching δ function at $t = 5$ in Fig.1a.

The width of $|\psi(Q, t)|^2$, i.e., $\sigma_Q(t)$, Brownian width $\sigma_\xi(t)$ and total width $\sigma_{Q\xi}(t)$, as function of t are also shown in Fig.2a.

The probability of falling into upper and lower potential well is shown in Fig.2b, i.e. they are constants starting from $t = 0_+$: for $\theta = \pi/4$, they are $\cos^2(\theta/2) = 0.8535$ (magenta); $\sin^2(\theta/2) = 0.1464$ (cyan). That is, they are predetermined. Their sum is 1(blue).

The profile of wave packets with spin up and down begins to separate at $t = 0.05$, as shown in Fig.3a. The wave packets at $t = 2$ are shown in Fig.3b, where we compare the profiles with and without Brownian motion when they are well separated.

The conclusion is: the probability distribution $\cos^2(\theta/2), \sin^2(\theta/2)$ is predetermined by the initial state $|\vec{a}\rangle$.

5. Implications for applying the result to the Bell theorem

We can now apply the results of Sections 1,2,3 to the Bell theorem. The main results of these sections are that the final result of the measurement is determined by the wave function of the initial state, as stressed at the end of Section 1, and clearly shown in Fig.2b. Since this is only for a specific model, the interpretation and the implication here are only suggestive.

5.1. We first repeat the statement of the Bell theorem to clarify the notations used here.

In the Bell theorem experiment [1,22], we assume that the two entangled particles leave the origin at point O and move in opposite directions, so that particle 1 reaches position A and measures $\sigma_1 \cdot \vec{a}$ and particle 2 reaches position B and measures $\sigma_2 \cdot \vec{b}$ (\vec{a}, \vec{b} are some unit vectors).

The paper assumed a hidden variable λ such that the result of measuring $\sigma_1 \cdot \vec{a}$ is determined by \vec{a} and λ as $A(\vec{a}, \lambda)$, and the result of measuring $\sigma_2 \cdot \vec{b}$ is determined by \vec{b} and λ as $B(\vec{b}, \lambda)$ in the same instance, and

$$A(\vec{a}, \lambda) = \pm 1; B(\vec{b}, \lambda) = \pm 1. \quad (23)$$

According to Bell, “The vital assumption is that the result B for particle 2 does not depend on the setting \vec{a} of the magnet for particle 1, nor A on \vec{b} ” [22]. Bell assumed that if $\rho(\lambda)$ is the probability distribution of λ , then the expectation value of the product of the two components $\sigma_1 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ is

$$P(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) \quad (24)$$

Bell supposes that the experimenter has a choice of settings for the second detector: it can be set either to \vec{b} or to \vec{c} . Bell proves that if Eq.(23) and Eq.(24) are correct, the difference in correlation between these two choices of detector settings must satisfy the inequality

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \leq 1 + P(\vec{b}, \vec{c})$$

Then, Bell shows it is easy to find situations where quantum mechanics violates the inequality, i.e., the “Bell inequality”, thus Eq.(23) and Eq.(24) derived from the “local hidden variable theory,” is incompatible with quantum mechanics.

5.2. The analysis of two separate measurements at remote positions A and B on two particles is equivalent to the analysis of two measurements on one particle at the origin

When we apply the result of Sections 1, 2, and 3, because the result is predetermined by the initial condition, it is clear that the measurement at any later time would not affect the initial probability amplitude. And, any conclusion from the result does not provide any information about whether the measurement at A influences B. Consequently, the experiment does not carry any information about “action at a distance,” or “spooky action,” or “nonlocal” property.

Therefore, we only need to compare the probability amplitude at time $t = 0$, i.e., we compare the measurement of $\sigma_1 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ at time $t = 0$. When we compare these two

measurements with the experiment in the Bell theorem, we can compare them at $t = 0$, which means we compare them at the origin O .

Furthermore, because the two entangled particles are completely correlated such that $\sigma_1 \cdot \vec{a} = -\sigma_2 \cdot \vec{a}$, we only need to compare $-\sigma_2 \cdot \vec{a}$ with $\sigma_2 \cdot \vec{b}$ at time $t = 0$ and investigate the condition for both to have definite values. In the following, when applying the results of Section 3, we can consider \vec{z} as \vec{b} , and take \vec{a} the same as the \vec{a} in Section 3.

5.3 The statement “There are no hidden variables for non-commuting observables to have definite values at the same time” is the consequence of the uncertainty principle. We discuss its implications for the Bell theorem and “Schrödinger’s cat problem”.

Because we start with $\sigma_1 \cdot \vec{a} = \pm 1$, and measure $\sigma_2 \cdot \vec{b}$, the complete correlation between particles 1 and 2 requires particle 2’s initial state to be either $\sigma_2 \cdot \vec{a} = 1$, or $\sigma_2 \cdot \vec{a} = -1$. So as long as $\sin \theta \neq 0$, particle 2’s initial state is not an eigenstate of $\sigma_2 \cdot \vec{b}$. Comparing with the Bell theorem, this is equivalent to measuring $\sigma_2 \cdot \vec{a}$ first and then immediately measuring $\sigma_2 \cdot \vec{b}$.

The initial state for the measurement of $\sigma_2 \cdot \vec{b}$ is then an eigenstate of $\sigma_2 \cdot \vec{a}$, not an eigenstate of $\sigma_2 \cdot \vec{b}$, even though it is a pure state, i.e., it is the result of the measurement of $\sigma_2 \cdot \vec{a}$. As a result, as long as $\sin \theta \neq 0$, the measurement of $\sigma_2 \cdot \vec{b}$ must have two possible outcomes. This is not determined by any external observer, but is because the initial state is not a definite state for the measurement of $\sigma_2 \cdot \vec{b}$, even though it is a definite state for $\sigma_2 \cdot \vec{a}$.

It is also clear that if $\sin \theta \neq 0$, there are no hidden variables that can give both $\sigma_2 \cdot \vec{a}$ and $\sigma_2 \cdot \vec{b}$ definite values. This is the Uncertainty Principle. Further, the probability of $\sigma_2 \cdot \vec{b} = 1$ is $\cos^2(\theta/2)$, and the probability of $\sigma_2 \cdot \vec{b} = -1$ is $\sin^2(\theta/2)$ according to Born rule.

In the present model case, it is very natural to understand why if $\sin \theta \neq 0$, there are two outcomes from $\sigma_2 \cdot \vec{b}$. Since the initial state is a pure state with \vec{a} on the Bloch sphere, it is a superposition of two components. The one with $\sigma_2 \cdot \vec{b} = 1$ will be deflected by the magnetic field to direction \vec{b} , the other with $\sigma_2 \cdot \vec{b} = -1$ will be deflected to direction $-\vec{b}$. This probability distribution between the two outcomes, on the other hand, exists from the start as a physical reality embodied later by magnetic deflection.

As a result, the measurement of $\sigma_2 \cdot \vec{a}$ is insufficient to prepare a definite state for the second measurement; we must still specify that the measurement setup in the environment after the start is still a measurement of $\sigma_2 \cdot \vec{a}$, or in other words, a measurement of $\sigma_2 \cdot \vec{b}$ but with $\vec{b} = \pm \vec{a}$.

The implication of this is that even though the initial wave function is the eigenvector of a complete set of commuting observables and has definite values, this alone cannot guarantee a definite status. Because, in addition to this, we must first make sure the measurement device is arranged to measure this same set of observables. From this point

of view, the wave function is not a complete description of the system; a description of the measurement environment must be included in order to have a complete description. One of the foundations of quantum mechanics, the Uncertainty Principle, is based on the fact that no physical system can be built to let two non-commuting observables have definite values; consequently, no one can prepare such an initial state in any experiment.

If we consider the case in Section 3 with $\vec{a} = \vec{x} = (1, 0, 0)$, then $\theta = \frac{\pi}{2}, \phi = 0$,

$$|\vec{a}\rangle = \cos(\theta/2)|+\rangle + e^{i\phi}\sin(\theta/2)|-\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

we may consider this as a model to compare with an extremely simplified “Schrödinger’s cat problem”. It is not an eigenstate of σ_z which is the measurement we are taking. According to our discussion above, the initial state is an eigenstate of the measurement of $\sigma \cdot \vec{a}$. It is not a definite state for the measurement of σ_z from the beginning, because it has a probability of 50% for spin up or down. So the initial measurement of spin $\sigma \cdot \vec{a}$ produced an ensemble of initial states for the following measurement of σ_z . We emphasize that this probability distribution is not generated at the end by someone observing it when the measurement of σ_z is completed, but is already determined right after the measurement of $\sigma \cdot \vec{a}$. The role of a final measurement of σ_z is to sort out the distribution, sample $|+\rangle$ or $|-\rangle$, and put them into the centers of the two split potential wells.

Therefore, there is no need to resort to any “external observer”.

No matter whether there is any one to observe the result, the probability distribution is certain from the very beginning. $|\vec{a}\rangle$ is called a “pure state” only in the sense that it will give a definite result if one measures $\sigma \cdot \vec{a}$. The main point is no one can prepare a definite state for σ_z by measuring $\sigma \cdot \vec{a}$.

5.4. At this point, one may ask why the Bell theorem sometimes seems to generate a sense of non-locality in quantum mechanics.

As we discussed above, the violation of Bell inequality by quantum mechanics is because the Uncertainty Principle contradicts the hidden-variable theory. It was unnecessary to replace the “hidden variable theory” with the “local hidden variable theory”. There is no need to resort to the word “local” in the derivation of Bell inequality.

According to reference [1], “In the words of physicist John Stewart Bell, for whom this family of results is named, ‘If [a hidden-variable theory] is local, it will not agree with quantum mechanics, and if it agrees with quantum mechanics, it will not be local’ ”.

This conclusion seems to generate a sense of some non-locality in quantum mechanics when the experiment indeed agrees with quantum mechanics. However, with a careful examination of the derivation in 4.1, as long as we assume there is a hidden variable theory,

i.e., Eq.(23) and Eq.(24), we can follow Bell's derivation and reach the same conclusion as the Bell theorem without resorting to locality or non-locality.

The the basic assumption by Bell for his derivation is “the result B for particle 2 does not depend on the setting \vec{a} of the magnet for particle 1, nor A on \vec{b} ”[22], as we referred to in 4.1. We emphasize here that communication between A and B (or not) is irrelevant.

Even if A does communicate with B such that B receives information in advance that the setting chosen by A is \vec{a} , as long as B does not choose the setting \vec{b} to be parallel to \vec{a} , Bell's assumption will still lead to Eq.(23) and Eq.(24), which then leads to Bell inequality without resorting to locality or non-locality, i.e., without resorting to whether there is any communication between A and B.

So we may rephrase the above statement (1) as a new statement (2): “If a hidden-variable theory can give definite values to non-commuting observables, the result will not agree with quantum mechanics, and if it agrees with quantum mechanics, then no hidden-variable theory can give definite values to non-commuting observables. “

As long as the Uncertainty Principle is correct, it is natural that quantum mechanics will violate Bell inequality, and also it rules out a “hidden variable theory”. Both the Bell theory and our discussion has nothing to do with whether there is non-locality, “action-at-a distance”, or not.

The theory we used is non-relativistic, which is justified because there is nothing close to light speed involved.

The comparison of statement (1) and statement (2) and the discussion above demonstrates the experiment's agreement with quantum mechanics does not provide information related to non-locality. The inclusion of the word "local" in Bell's statement here shifts the emphasis from "hidden variables theory" to "local hidden variable theory". This contradicts quantum mechanics, giving the impression that the confirmation of quantum mechanics prediction leads to some non-locality. Hence, the discussion above suggests this statement may be misleading if one does not carefully examine the basic ground underlying the derivation of the Bell theorem.

We conclude that as long as the Uncertainty Principle is established, violation of the Bell inequality in the experiment entirely ruled out the existence of any hidden variable theory, irrelevant of any locality or non-locality.

6. Summary: The implications for the understanding of the statistical interpretation of quantum mechanics

The preceding discussion in Section 4 appears to be simply repeating the fundamentals of quantum mechanics; thus, as quantum mechanics predicts, it leads to the violation of Bell Inequality. Indeed, we have demonstrated that in quantum mechanics, when the Born Rule is applied to the initial state, the result is a self-consistent theory that includes using

the Schrödinger equation to describe the quantum measurement process itself, without resorting to non-locality, hidden variable, or any external observer; without resorting to whether the measurement at A immediately influences the measurement at B or the distance between A and B. Thus, it removes many mysteries surrounding the statistical interpretation of quantum mechanics.

We emphasize that because the discussion is only for a specific model, the implications of the discussion in Section 4 are only suggestive and summarized as:

1. The derivation of the wave function collapse in the measurement process in Sections 1, 2, and 3 shows the probability distribution is determined from the very beginning (i.e., $t = 0$), not at the end.
2. As a result, when Bell inequality is violated in an experimental test of the Bell theorem, there is no paradox arising from the concepts of “non-locality” or “action at a distance”.
3. A linear superposition of the eigenstates of a complete set of commuting observables is an ensemble of systems with a probability distribution determined by the Born rule. A wave function will give a definite measurement result only if it is the eigenstate of the observables being measured. This crucial point removes many mysteries surrounding the statistical interpretation of quantum mechanics.
4. This suggests that as long as the Born rule is applied to the interpretation of the initial state, the Born rule at the end of the measurement can be derived from the Schrödinger equation.

It is important to note, in the model case we have discussed here, 1) the correlation between the two entangled particles is assumed to be completely preserved, and 2) the fact that the discussion about the violation of Bell inequality without resorting to the distance between A and B does not imply that the distance between A and B is irrelevant to the experiment on the entangled particles. On the contrary, the longer distance between A and B means the experiment design needs to preserve the correlation between the two particles over a longer distance.

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Appendix I

The following expressions are used in the derivation in Sections 1, 2, and 3. In Eq.(7). with $a_1(t) = \frac{\mu e^{-\nu t} - \nu e^{-\mu t}}{\mu - \nu}$, $a_2(t) = -\frac{e^{-\mu t} - e^{-\nu t}}{\mu - \nu}$, $\mu \equiv \frac{\eta}{2} + \omega i$, $\nu \equiv \frac{\eta}{2} - \omega i$ and here $\omega \equiv \left(\omega_0^2 - \frac{\eta^2}{4}\right)^{1/2}$, we have

$$\begin{aligned} a_1(t) &= e^{-\frac{\eta}{2}t} \left(\cos(\omega t) + \frac{\eta}{2\omega} \sin(\omega t) \right) \\ a_2(t) &= \frac{1}{\omega} e^{-\frac{\eta}{2}t} \sin(\omega t) \\ \dot{a}_1(t) &= -\omega_0^2 a_2(t) \\ \dot{a}_2(t) &= e^{-\frac{\eta}{2}t} \cos(\omega t) - \frac{\eta}{2\omega} e^{-\frac{\eta}{2}t} \sin(\omega t) \\ a_1(t)\dot{a}_2(t) - \dot{a}_1(t)a_2(t) &= e^{-\eta t} \end{aligned}$$

where we can confirm that $a_1(0) = 1, a_2(0) = 0, \dot{a}_1(0) = 0, \dot{a}_2(0) = 1$.

Then, use Eq.(9) and the expressions of $a_1(t), a_{12}(t), \dot{a}_1(t), \dot{a}_2(t)$, we find the commutator

$$[Q, \dot{Q}] = [a_1(t)Q_0 + a_2(t)\dot{Q}_0, \dot{a}_1(t)Q_0 + \dot{a}_2(t)\dot{Q}_0] = e^{-\eta t} \frac{i\hbar}{M}$$

Also in the solution of the damped oscillator equation Eq.(5), i.e., Eq.(7), the coefficients of the j 'th bath oscillator contribution to q

$$\begin{aligned} b_{j1}(t) &= -\frac{c_j}{M} \text{InverseLaplace}\left(\frac{s}{(s^2 + \omega_j^2)(s + \mu)(s + \nu)}\right) \\ &= -\frac{c_j}{M} \left[\frac{\omega_j(\mu + \nu) \sin(t\omega_j)}{(\mu^2 + \omega_j^2)(\nu^2 + \omega_j^2)} + \frac{(\mu\nu - \omega_j^2) \cos(t\omega_j)}{(\mu^2 + \omega_j^2)(\nu^2 + \omega_j^2)} + \frac{\mu e^{-\mu t}}{(\mu - \nu)(\mu^2 + \omega_j^2)} - \frac{\nu e^{-\nu t}}{(\mu - \nu)(\nu^2 + \omega_j^2)} \right] \\ b_{j2}(t) &= -\frac{c_j}{M} \text{InverseLaplace}\left(\frac{1}{(s^2 + \omega_j^2)(s + \mu)(s + \nu)}\right) \\ &= -\frac{c_j}{M} \left[\frac{(\mu\nu - \omega_j^2) \sin(t\omega_j)}{\omega_j(\mu^2 + \omega_j^2)(\nu^2 + \omega_j^2)} - \frac{(\mu + \nu) \cos(t\omega_j)}{(\mu^2 + \omega_j^2)(\nu^2 + \omega_j^2)} - \frac{e^{\mu(-t)}}{(\mu - \nu)(\mu^2 + \omega_j^2)} + \frac{e^{\nu(-t)}}{(\mu - \nu)(\nu^2 + \omega_j^2)} \right] \\ b_{j1}(t \rightarrow \infty) &= -\frac{c_j}{M} \left[\frac{\omega_j(\mu + \nu) \sin(t\omega_j)}{(\omega_j^2 + \mu^2)(\omega_j^2 + \nu^2)} + \frac{(\mu\nu - \omega_j^2) \cos(t\omega_j)}{(\omega_j^2 + \mu^2)(\omega_j^2 + \nu^2)} \right] \\ \omega_j b_{j2}(t \rightarrow \infty) &= -\frac{c_j}{M} \left[\frac{(\mu\nu - \omega_j^2) \sin(t\omega_j)}{(\omega_j^2 + \mu^2)(\omega_j^2 + \nu^2)} - \frac{\omega_j(\mu + \nu) \cos(t\omega_j)}{(\omega_j^2 + \mu^2)(\omega_j^2 + \nu^2)} \right] \\ b_{j1}(t \rightarrow \infty)^2 + \omega_j^2 b_{j2}(t \rightarrow \infty)^2 &= \frac{c_j^2}{M^2} \frac{1}{(\omega_0^2 - \omega_j^2)^2 + \eta^2 \omega_j^2} \end{aligned}$$

where we can confirm that $b_{j1}(0) = b_{j1}(0) = \dot{b}_{j1}(0) = \dot{b}_{j1}(0) = 0$.

Appendix II

Denote the eigenvector of $Q(t) = a_1(t)Q_0 + a_2(t)\dot{Q}_0$ with eigenvalue Q_1 as $u_{Q_1}(Q_0, t)$, where $Q_0 \equiv q_0 - \sigma_z d$, $\dot{Q}_0 \equiv -\frac{i\hbar}{M} \frac{\partial}{\partial q_0} \equiv -\frac{i\hbar}{M} \frac{\partial}{\partial Q_0}$, $Q(t)$ and $\dot{Q}(t)$ are understood as a function of $q_0 - \sigma_z d$, \dot{q}_0 from Eq.(9). Since σ_z commute with Q_0 , \dot{Q}_0 , there is no ambiguity for the functions of Q_0 , \dot{Q}_0 regarding the order of σ_z in any product in the function. The eigenequation

$$\left(a_1(t)Q_0 - \frac{i\hbar a_2}{M} \frac{\partial}{\partial Q_0} \right) u_{Q_1} = Q_1 u_{Q_1}$$

leads to

$$u_{Q_1} = C(Q_1, t) \exp \left[-i \frac{M}{2\hbar a_2} (a_1 Q_0^2 - 2Q_1 Q_0) \right]$$

$C(Q_1, t)$ is determined by orthonormal condition on u_Q , i.e., $\int dQ_0 u_{Q_2}^*(Q_0) u_{Q_1}(Q_0) = \delta(Q_1 - Q_2)$ as $C(Q_1, t) = \left(\frac{M}{2\pi\hbar a_2} \right)^{\frac{1}{2}} e^{-i \frac{M}{2\pi\hbar a_2} \phi(Q_1, t)}$ in Eq.(10).

Appendix III Wave function probability density with Brownian motion

We abbreviate the wave function density as $f(\sum_j \xi_{j1}) = |\psi(q_1 - s_1 d - \sum_j \xi_{j1}, s_1, t)|^2$, then using the Taylor expansion of f it is straight forward to show that

$$\begin{aligned} & \int \int \dots \int \int f(\sum_j \xi_j) \prod_j |\chi_j(\xi_j, t)|^2 d\xi_j \\ &= \frac{1}{\prod_j \sqrt{2\pi\sigma_{\xi_j}}} \int \int \dots \int \int f(\sum_j \xi_j) \exp \left(-\sum_j \frac{\xi_j^2}{2\sigma_{\xi_j}^2} \right) \prod_j d\xi_j \quad (25) \\ &= \frac{1}{\sqrt{2\pi\sigma_\xi}} \int f(\xi) \exp \left(-\frac{\xi^2}{2\sigma_\xi^2} \right) d\xi \\ &\sigma_\xi^2 = \sum_j \sigma_{\xi_j}^2 \end{aligned}$$

Thus

$$\begin{aligned} |\Psi(q, s, t)|^2 &\equiv \int \int \dots \int \int |\Psi(q, s, \{\xi_j\}, t)|^2 \prod_j d\xi_j \\ &= \int \int \dots \int \int |\psi(q - sd - \sum_j \xi_j, s, t)|^2 \prod_j |\chi_j(\xi_j, t)|^2 d\xi_j \quad (26) \\ &= \frac{1}{\sqrt{2\pi\sigma_\xi}} \int |\psi(q - sd - \xi, s, t)|^2 \exp \left(-\frac{\xi^2}{2\sigma_\xi^2} \right) d\xi \end{aligned}$$

For probability density of Eq.(22),with $|\psi(Q, t)|^2 = |\psi(q - d, t)|^2$ in Eq.(19) we have

$$\begin{aligned} |\Psi_+(q, t)|^2 &= \cos^2(\theta/2) \frac{1}{\sqrt{2\pi}\sigma_\xi} \int |\psi(q - d - \xi, 1, t)|^2 \exp\left(-\frac{\xi^2}{2\sigma_\xi^2}\right) d\xi \\ &= \frac{1}{\sqrt{2\pi}\sigma_\xi} \int \left(\frac{1}{2\pi\sigma_Q^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(q - sd - \xi - a_1z)^2}{2\sigma_Q^2}\right) \exp\left(-\frac{\xi^2}{2\sigma_\xi^2}\right) d\xi \\ &= \cos^2(\theta/2) \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{Q\xi}} \exp\left(-\frac{(q - d + a_1d)^2}{2\sigma_{Q\xi}^2}\right) \end{aligned}$$

where the width of the distribution is $\sigma_{Q\xi}^2 = \sigma_Q^2 + \sigma_\xi^2$. The same way the probability of the lower potential well with spin down is

$$|\Psi_-(q, t)|^2 = \sin^2(\theta/2) \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_{Q\xi}} \exp\left(-\frac{(q + d - a_1d)^2}{2\sigma_{Q\xi}^2}\right)$$

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