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*Original Paper*

# Probabilistic Bohmian Quantum Mechanics and Non-standard Analysis

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**Abstract:** In this paper, by using a hyperfinite dimensional space and time lattice ($\mathcal{ST}$-**lattice**) of nonstandard analysis, we present a variant of Bohmian interpretation of quantum mechanics which we call *probabilistic Bohmian mechanics*, PBM. We describe the model for non-relativistic quantum mechanics in detail and elaborate on its relativistic extension. The assumption of *quantum equilibrium* does not exist in the PBM model and its absence is compensated for by assuming that the particles are moving with infinite speed on a space time lattice according to probability density, which in PBM is regarded as density of space and time *position states*. In relativistic extension the PBM model assumes that Lorentz symmetry of space and time is just a wave phenomenon related to the space time symmetry of wave equations on real standard axes, whereas the motion of particles with infinite speed can only be described within the hyperfinite dimensional time and space lattice of preferred space time foliation ($\mathcal{PST}$-**lattice**). In the PBM model the particle trajectories do not exist on the standard real time and space axes of any Lorentz frame and thus the assumption of wave function collapse is necessary. The wave function collapse is considered as a two-step process of decoherence (on standard real axes) and the subsequent particle jump (on the $\mathcal{ST}$-**lattice**). For the PBM model, as an *objective collapse theory*, the causality problem, related to wave function collapse, is addressed by the preferred space-time foliation.

**Keywords:** Interpretations of quantum mechanics; Ontological wave function collapse theories; Bohmian quantum mechanics; Born rule; Mathematical foundation of quantum mechanics; Non-standard analysis

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## 1. Introduction

It is now nearly one hundred years since the initiation of modern quantum mechanics (and later its extension, the quantum field theory), but despite this and its undeniable success in calculating observable phenomena, the solution to its measurement problem, its non-locality and the true nature of its wave-particle duality are still unsettled issues in the physics community. So far, we have had different interpretations for the solution of the measurement problem (e.g. many wor1ds interpretation, objective collapse theories, etc.), all of which deal with different problematic issues and which for some people are found not to be fully compelling [31].

Any quantum interpretation which is built upon the assumption that both particles (with space-time trajectories) and waves are real objects can be called, a *Bohm like interpretation*. One known example in particular is Bohmian mechanics, $\mathcal{BM}$ ([7], [8]. [5] and [29]) which relates best with the deterministic world of non-relativistic large objects, by introducing the guiding equation, $\frac{d(Q_k(t))}{dt} = \frac{\hbar}{m_k}\mathrm{Im}\frac{\psi^*\nabla_k\psi}{\psi^*\psi}(Q(t),t)$, although to justify statistics of quantum mechanics (Born rule) for small particles, in addition to the guiding equation and the Schrödinger wave equation, it is necessary to accept the *quantum equilibrium hypothesis*, which asserts that we always have initial $|\psi|^2$ distribution of particle configurations in an ensemble. In fact, the quantum equilibrium hypothesis is too big an assumption, of obscure origin, which becomes inevitable if, one accepts that the guiding equations are describing particle motions and trajectories. As we argue below, without the quantum equilibrium hypothesis, any *Bohm like interpretations* of quantum mechanics, which assumes finite speed for particles (whether they move deterministically or probabilistically), is inconsistent with Born rules and thus it is no wonder that no such Bohm like interpretation of $QM$ (without the quantum equilibrium hypothesis) has been presented before. One other issue with Bohmian mechanics, from an ontological perspective, is the role and the meaning of the wave function. Bohmian mechanics, which is deterministic in the non-relativistic domain by Bell type extensions, becomes a combination of deterministic (guiding wave equations) and probabilistic (creation and annihilation of particles) processes in which the wave function has a determining role even though we do not have a clear picture of what the wave function really is. In other words, $\mathcal{BM}$ makes the meaning and practical use of wave function, which is totally probabilistic in common usage of $QM$ (Born's rule), more complicated without adding to its ontology. It should also be mentioned that one of the problems which exists today in the extension of Bohmian mechanics to the relativistic domain, is finding a proper guiding equation for Photons [29]. Despite all of this, one should also consider the fact that any kind of a Bohm-like model which relates particle motions to their wave function and their positions at each instant of time, will require a preferred time foliation for relativistic extension.

Furthermore, to recover the results of field theory and to consider particle creation and annihilation, the Bell type extensions for Bohm like mechanics, are possibly the way forward.

The goal of this paper is to present a probabilistic ontological model without adopting the quantum equilibrium hypothesis. To begin the discussion and explain the relationship between this goal and non-standard analysis, we can consider wave function broadening of free particles in $QM$. Let us suppose the wave function of a one particle system is confined in the sphere with the radius $R_1$ (the wave function, $\psi$, is zero out side the sphere) and, in addition, let us consider the ensemble distribution of particles is concentrated in a smaller sphere with the radius $R_2$ ($R_1 > R_2$) with the same center point (the assumption which is against quantum equilibrium hypotheses). For the Schrödinger equation (non-relativistic $QM$) the tail speed of wave function is infinite, so obviously no matter how the particle moves (whether it is probabilistic or deterministic), with finite speed, one can not justify Born's statistical rule. The only option we have is to consider particles of infinite speed and one of the known logical framework in which this idea can be raised is Non-standard analysis (NSA). Even if, in the relativistic domain, one assumes such a localized wave packet (which is a combination of positive and negative particle states) is spreading with speed $C$ and particles in ensemble (confined within smaller sphere) are moving with the same speed $C$, there is always a distance $R_1 - R_2$, from the edge of the wave function, untouched by particles in the ensemble. The only other possibility is that one assumes the particles are moving with speed $V >> C$ and thus the particles at the edge of the smaller sphere in the ensemble will reach the edge of wave function after time $t = \frac{R_1 - R_2}{V - C}$ and the speed $V$ can be considered so large that it can justify Born's rule within our experimental accuracy. However, it should be mentioned that from a theoretical point of view the fact that $t$ is non-zero is not consistent with Born's rule and also both speeds much faster than light and infinite speed, confront the problem of inconsistency with special relativity where the choice of infinite speed is more capable in coping with the issue.

The history of non-standard analysis is full of debate. It was first Archimedes, who postulated, the Archimedes axiom for numbers, in contradiction to the existence of infinite and infinitesimal numbers, although he used infinitesimal as a useful tool for derivations (but not for proofs). The idea of infinitesimals appeared again in the works of the pioneers of calculus, Newton and Leibnitz, but due to insufficient logical grounds, these ideas were soon replaced by the methods of Bolzano and Cauchy. The idea was reborn again in the early sixties of the twentieth century by Abraham Robinson who introduced hyperreal, [25], as the first examples of Non-Archimedean real closed field of numbers (which for example can be constructed on the mathematical grounds of ZFC set theory). In this paper we take an approach to the foundation and interpretation of quantum mechanics based on hyperfinite dimensional spaces and also hyperfinite splitting of the time axis of nonstandard analysis (NSA). For the purpose of clarification of mathematical notation the following points should be noted. In NSA, the field of reals, $\mathbb{R}$ (called standard real numbers) is extended to $^*\mathbb{R}$ that includes bounded numbers (elements of the set $\mathcal{B}$) which are not infinitely large (we denote $l \in \mathcal{B}$ by $l \ll \infty$), infinitely large numbers (element of the set $^*\mathbb{R} \setminus \mathcal{B}$) , which are greater than any real number (we denote $\Omega \in {}^*\mathbb{R} \setminus \mathcal{B}$ by $\Omega \sim \infty$) and their reciprocals ($\epsilon = 1/\Omega$) which are called infinitesimal numbers (elements of the set $\mathcal{J}$), the absolute values of which are smaller than any positive standard numbers (we denote $\epsilon \in \mathcal{J}$ by $\epsilon \approx 0$). Two bounded numbers, $l_1$ and $l_2$, are called infinitely close, if $(l_1 - l_2) \in \mathcal{J}$ and it is denoted by $l_1 \approx l_2$. Any bounded number, $l$, is infinitesimally close to a real number (called its standard part): $l \approx \mathbf{st}(l)$ where $\mathbf{st}(l) \in \mathbb{R}$. Also in NSA the set of natural numbers $\mathbb{N}$ is extended to $^*\mathbb{N}$ which includes natural numbers (elements of $\mathbb{N}$) which are called finite hyperinteger and infinite hyperinteger numbers (elements of $^*\mathbb{N} \setminus \mathbb{N}$) which are greater than any natural number. A vector space whose dimension is in $^*\mathbb{N}$ is called hyperfinite dimensional space. The application of NSA to the foundation of quantum mechanics has been carried out by some authors (e.g. [13], [1], [2], [24], [6]) mainly with the original motivation of presenting an alternative approach to von Neumann axioms of *the mathematical foundation of quantum mechanics* (describing observable and self-adjoint operators). Our work here is different in several ways, the most important of which is that, in the application of NSA our focus is to build a construction by which we present a new interpretation of quantum mechanics. In particular, we not only make the space but also the time axis discrete by infinitesimal spacing such that, for example, in non-relativistic $QM$ our starting point should be the time dependent Schrödinger equation rather than the time independent Schrödinger equation.

Section 2 is about PBM interpretation for non-relativistic $QM$. Here, we not only make the space but also the time axis discrete. We assume there is the smallest time interval $\Delta t \approx 0$ in nature and for each particle of mass $m$ there is the smallest length, $L \approx 0$ (given by (1)). We define hyperfinite dimensional time and space lattice, $\mathcal{ST}$-**lattice** (definition (3)) by using these infinitesimal values. Our fundamental principle in extension to non-standard numbers is to have complete correspondence with $QM$ results on standard real space and time axes. Therefore, in order to express wave equation, we do not require ourselves to maintain Galilean symmetry (for Schrödinger equation) or Lorentzian symmetry (for Dirac equation) on non-standard axes, rather, it is sufficient to show such symmetries exist on real standard axes. $\mathcal{ST}$-**lattice** is used here to describe two things; first, the locations (of mass centers) of several particles, $\hat{x}$, at specific time $\hat{t}$ and secondly, the extended wave function, $\widehat{\psi}(\hat{x}, \hat{t}) = \sqrt{\widehat{R}(\hat{x}, \hat{t})} \exp(i\,\widehat{\theta}(\hat{x}, \hat{t}))$, describing those particles. In this paper, we interpret the wave function square, $\widehat{R}(\hat{x}, \hat{t})$, as the *density of position states* on which the mass centers of a particle can rest at each position $\hat{x}$ in the time interval $[\hat{t}, \hat{t} + \Delta t]$. The motions of particles are described by *probabilistic jumps* , for which different models can be presented. One of the simplest of such models which leads to Born's statistical rule is introduced by (15) and (21). As with $\mathcal{BM}$, the description of particle motions by probabilities (21) in the PBM model is non-local, for example the probability for motion of a particle to any neighboring points depends on the position of other particles described by the wave function. The particles do have trajectories on the $\mathcal{ST}$-$\mathbf{lattice}$ but not on corresponding standard real space and time axes. This necessitates the assumption of the wave function collapse, as is further discussed in part 2.3. The wave function collapse is described as a two step process, the wave function decoherence and the subsequent particle jump (given by the probability equation (17)).

In relativistic extension, as mentioned earlier, the same problem of wave function broadening exists, leading us again to assume infinite speed for particles. The motion of particles with infinite speed encounters two problems; firstly, the special relativity and secondly, the measurement problem. As discussed in section 3, in the PBM model Lorentz symmetry is only related to wave function equations and not the particle motions. The particles are assumed to move with infinite speed within the hyperfinite dimensional space and time lattice of the preferred Lorentz frame (which we call $\mathcal{PST}$-$\mathbf{lattice}$) filling all possible position states at every instant of standard real time $t$ and what any Lorentz frame perceives (including preferred frame of reference) at each space and time moment on a standard real time axes is just the wave function. In another words, the Lorentz symmetry of space and time, according to which all our measuring devices work, the rest mass of particles and mass increase by velocity etc, are just wave phenomena. These considerations bring up the second problem, which is the measurement problem. As will be discussed in the PBM model, the interpretation of wave function as something which represents position states, along with the assumption of infinite speed for particles, makes the wave function collapse necessary for the model. In other words the PBM model is a type of objective wave collapse theory. As is discussed in section 3, the PBM model as an objective collapse theory, offers a solution for the causality problem of such models by using preferred time foliation. In section 4, the PBM is compared with some other interpretations of $QM$. Finally a number of related subjects are included in Appendices A and B.

## 2. PBM interpretation for non-relativistic $QM$

*2.1. Hyperfinite dimensional space-time lattice* Let us assume $L \approx 0$ is the smallest

distance for motion of a particle of mass $m$ and begin with this question: What is the maximum time the center of mass of a particle can stay in distance $L$, according to the uncertainty principle? The answer is,

$$\Delta p\,\Delta x = (\frac{mL}{\tau})\,L \geq \hbar \quad \Longrightarrow \quad \tau \leq \frac{m\,L^2}{\hbar},$$

and thus the maximum time is $\Delta t = m\,L^2/\hbar$. It should be mentioned that if such a jump (with $\tau = \Delta t$) takes place, the speed of the particle would be $V_{jump} = \hbar/mL \sim \infty$, as will be discussed in part 2.3. Therefore consistent with uncertainty principle we define the following relationship as,

$$L \approx 0, \qquad\qquad (\Delta t) = \frac{m\,L^2}{\hbar}, \tag{1}$$

between smallest time $\Delta t$ ≈0 and smallest length $L \approx 0$ for the motion of particle of mass $m$. Let us first define space and time lattice for motion of one particle with the mass $m$ in one dimension. We assume there is a non-zero positive real number, $l_0$ with length dimension and $N^*$ is infinite hyperinteger number ($N^* \in {}^*\mathbb{N} \setminus \mathbb{N}$) such that,

$$L = \frac{l_0}{N^*}. \approx 0.$$

Now by taking $M^* = (N^*)^2$ we can make the space line discrete as $-(M^*)\,L$,$\cdots$,$-L$,0,$L$,$\cdots$, $(M^*)L$. The end points $\pm(M^*)\,L = \pm N^* l_0$ are infinite. Thus hyperfinite one dimensional space axis is given by

$$\{-(N^*)\,l_0, \cdots, -L, 0, L, \cdots, (N^*)l_0\} \equiv [\mathbb{Z}_{2(M^*)+1}]_{space}.$$

The length of a whole space line is, $(2N^*)\,l_0$. Now we can rewrite $(\Delta t)$ as

$$(\Delta t) = \frac{m\,L^2}{\hbar} = \frac{(\Delta t_0)}{(N^*)^2} \quad \text{where,} \quad (\Delta t_0) = \frac{m\,{l_0}^2}{\hbar}.$$

By taking $M_1^* = (N^*)^3$, the time axis can be made discrete as $-(M_1^*)\Delta t$ $, \cdots, -\Delta t, 0, \Delta t, \cdots, (M_1^*)\,\Delta t$. The end points $\pm M_1^* \Delta t = \pm N^* \Delta t_0$ are infinite. Thus hyperfinite time axis is given by

$$\{-N^*\Delta t_0, \cdots, -\frac{m\,L^2}{\hbar}, 0, \frac{m\,L^2}{\hbar}, \cdots, N^*\Delta t_0\} \equiv [\mathbb{Z}_{2(M_1^*)+1}]_{time}.$$

The length of the time line is therefore equal to, $2N^*\Delta t_0$. To define the space and time lattice of $n$ particles in 3 dimensional space, with masses $m_1, m_2, ..., m_n$, the infinitesimal $\Delta t$ should be equal for all particles. Thus, configuration space can become discrete as follows,

$$L_i \approx 0, \qquad (\Delta t) = \frac{m_i\,L_i^2}{\hbar}, \tag{2}$$
$$m_1 {L_1}^2 = m_2 {L_2}^2 = ... = m_n {L_n}^2.$$

In the similar way to one dimensional case of one particle, we can assume there are $n$ real positive numbers $l_i$ for $i = 1, ..., n$, with length dimensions such that we have,

$$L_i = \frac{l_i}{N^*}, \qquad i = 0, 1, 2, ..., n,$$

where $l_i$'s satisfy, $m_1\, {l_1}^2 = m_2\, {l_2}^2 ... = m_n\, {l_n}^2$. By defining the space line for each particle $m_i$,

$$[\mathbb{Z}_{2(M^*)+1}]_{i\,-\,space} \equiv \{-(N^*)\, l_i, \cdots, -L_i, 0, L_i, \cdots, (N^*) l_i\} \qquad i = 0, 1, 2, ..., n,$$

the configuration space for $n$ particles is given by,

$$\Pi_{i=1}^{n} \left([\mathbb{Z}_{2(M^*)+1}]_{i-space}\right)^3.$$

Therefore the space and time lattice, $\mathcal{ST}$-**lattice** is defined as,

$$\mathcal{ST}\textbf{-lattice} \quad \equiv \Pi_{i=1}^{n} \left([\mathbb{Z}_{2(M^*)+1}]_{i-space}\right)^3 \times [\mathbb{Z}_{2(M_1^*)+1}]_{time}. \tag{3}$$

In this paper we write position variables and functions on the $\mathcal{ST}$-**lattice** using characters with a hat, for example $\hat{x}, \hat{y}, \hat{z}, ...$ (for space variables) and $\hat{t}$, $\hat{t}'$ (for time variables) versus standard reals $x$,$y$,$z$ and $t$,$t'$. By location $\hat{x}$ and time $\hat{t}$ we mean spatial interval $[\hat{x}, \hat{x}+L]$ and time interval $[\hat{t}, \hat{t}+\Delta t]$. Although a standard real number may or may not be exactly on the $\mathcal{ST}$-**lattice** , nevertheless, it can be considered as a generalization to standard real axes because it contains points which are infinitely close to any standard real numbers (e.g. equal up to all decimal digits) and there are ways to attribute to any real number a unique point on the hyperfinite dimensional lattice. In this paper we sometimes refer to the $\mathcal{ST}$-**lattice** as the hyperfinite dimensional space and time lattice or just the hyperreal axes (or lines). Due to existence theorems in NSA, the extension of $\psi(x)$ (defined on standard real space line) to $\widehat{\psi}(\hat{x})$ (defined on hyperfinite dimensional lattice $[\mathbb{Z}_{2(M^*)+1}]_{space}$) is expected to exist and satisfy mathematical conditions, such as continuity,

$$\forall \hat{x} \in [\mathbb{Z}_{2(M^*)+1}]_{space}, \qquad \mathbf{st}\widehat{\psi}(\hat{x}) = \psi(\mathbf{st}(\hat{x})),$$

and also the condition for having derivative,

$$\forall r \in \mathbb{Z}, \qquad \mathbf{st}(\frac{\widehat{\psi}(\hat{x} + r\,L) - \widehat{\psi}(\hat{x})}{r\,L}) = \partial_x \psi(x)|_{x=\mathbf{st}(\hat{x})},$$

whenever and wherever needed. .

*2.2. Wave equations on hyperreal lines*

In this section we want to express the Schrödinger equation on the $\mathcal{ST}$-**lattice**. We also partly explain what the wave function is. When taking into consideration the assumption of the existence of the smallest length and time, it seems logical to express the wave equation using a finite difference formula on the $\mathcal{ST}$-**lattice**. Here, for simplicity, we consider the one particle spinless Schrödinger equation in one dimension

$$\begin{cases} \partial_t\psi(x,t) = i\frac{\hbar}{2m}\partial_x^2\,\psi(x,t) - \frac{i}{\hbar}v(x)\psi(x,t), & (x\in\mathbb{R},\;\; t>0), \\ \psi(x,0) = \zeta(x), & (x\in\mathbb{R}). \end{cases} \tag{4}$$

and its wave function, $\psi(x,t) = \sqrt{R(x,t)}\exp\left(i\,\theta(x,t)\right)$ but we generalize to the case of the 3 dimensional $n$-particle Schrödinger equation in section 2.4. We begin by making the differential equation (4) discrete on the $\mathcal{ST}$-**lattice**. The initial wave function is assumed to be square integrable, $\int_{-\infty}^{\infty} dx\zeta^*(x)\,\zeta(x) = 1$, with $\partial_x\zeta(\pm\infty) = 0$. We can extend $\zeta(x)$ to $\mathcal{ST}$-**lattice**, $\widehat{\zeta}(\hat{x})$ and demanding,

$$\sum_{\hat{x}\in[\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} L\;\widehat{\zeta}(\hat{x})\widehat{\zeta}^*(\hat{x}) \;\; = \;\; 1 \tag{5}$$

Using the relation (1) between $\Delta t$ and $L$ and finite difference formulas, we can express the Schrödinger equation on the hyperfinite dimensional space and time lattice as a simple linear recursive formula,

$$\widehat{\psi}(\hat{x},\hat{t}+\Delta t) = \widehat{\psi}(\hat{x},\hat{t}) - i\,\widehat{\psi}(\hat{x},\hat{t}) + \frac{i}{2}\big(\widehat{\psi}(\hat{x}-L,\hat{t}) + \widehat{\psi}(\hat{x}+L,\hat{t})\big) - \frac{i}{\hbar}\widehat{v}(\hat{x})\widehat{\psi}(\hat{x},\hat{t})\Delta t \tag{6}$$

It can be easily checked that the above equation, on standard real time and space lines reads,

$$\begin{aligned} \partial_t\psi(x,t) &= \mathbf{st}[(\frac{\widehat{\psi}(\hat{x},\hat{t}+\Delta t) - \widehat{\psi}(\hat{x},\hat{t})}{\Delta t})] \\ &= \mathbf{st}[\frac{i\hbar}{2mL^2}\big(\widehat{\psi}(\hat{x}-L,\hat{t}) + \widehat{\psi}(\hat{x}+L,\hat{t}) - 2\widehat{\psi}(\hat{x},\hat{t})\big) - \frac{i}{\hbar}\widehat{v}(\hat{x})\widehat{\psi}(\hat{x},\hat{t})] \\ &= i\frac{\hbar}{2m}\partial_x^2\,\psi(x,t) - \frac{i}{\hbar}v(x)\psi(x,t), \end{aligned}$$

where $x = \mathbf{st}(\hat{x})$ and $t = \mathbf{st}(\hat{t})$. By using equation (6) one can check how fast the tail of a free wave packet ($v(x) = 0$), initially zero outside a bounded region of space, would spread on the hyperfinite dimensional lattice (the tail speed). For Schrödinger equation the tail speed is, $L/\Delta t = \hbar/(mL) \sim \infty$.

Expressing $J(x,t) = \frac{\hbar}{2\,m\,i}(\psi^*(x,t)\partial_x\psi(x,t) - \psi(x,t)\partial_x\psi^*(x,t))$, the probability current on the $\mathcal{ST}$-**lattice** we have,

$$\widehat{J}(\hat{x},\hat{t}) = \frac{i\hbar}{2mL}\Big(\widehat{\psi}(\hat{x},\hat{t})\big(\widehat{\psi}^*(\hat{x}-L,\hat{t}) - \widehat{\psi}^*(\hat{x},\hat{t})\big) - \widehat{\psi}^*(\hat{x},\hat{t})\big(\widehat{\psi}(x-L,\hat{t}) - \widehat{\psi}(\hat{x},\hat{t})\big)\Big). \tag{7}$$

Now by writing the change in $\widehat{R}(\hat{x},\hat{t}) = \widehat{\psi}(\hat{x},\hat{t})\widehat{\psi}^*(\hat{x},\hat{t})$, at location $\hat{x}$ from time $\hat{t}$ to $\hat{t}+\Delta t$,

$$\Delta\widehat{R}(\hat{x},\hat{t}) = \widehat{\psi}(\hat{x},\hat{t}+\Delta t)\ \widehat{\psi}^*(\hat{x},\hat{t}+\Delta t) - \widehat{\psi}(\hat{x},\hat{t})\widehat{\psi}^*(\hat{x},\hat{t}),$$

and using (6) and (7) we get,

$$\frac{\Delta\widehat{R}(\hat{x},\hat{t})}{\Delta t} = \frac{\widehat{J}(\hat{x},\hat{t}) - \widehat{J}(\hat{x}+L,\hat{t})}{L} + \frac{\widehat{A}_l(\hat{x},\hat{t})}{\Delta t}, \tag{8}$$

where, $\widehat{A_l}(\hat{x},\hat{t}) = \widehat{\psi}(\hat{x},\hat{t})\widehat{\psi^*}(\hat{x},\hat{t}) - \frac{1}{2}\widehat{\psi}^*(\hat{x},\hat{t})\big(\widehat{\psi}(\hat{x}-L,\hat{t}) + \widehat{\psi}(\hat{x}+L,\hat{t})\big) - \frac{1}{2}\widehat{\psi}(\hat{x},\hat{t})\big(\widehat{\psi}^*(\hat{x}-L,\hat{t}) + \widehat{\psi}^*(\hat{x}+L,\hat{t})\big) + \frac{1}{4}\big(\widehat{\psi}^*(\hat{x}-L,\hat{t}) + \widehat{\psi}^*(\hat{x}+L,\hat{t})\big)\big(\widehat{\psi}(\hat{x}-L,\hat{t}) + \widehat{\psi}(\hat{x}+L,\hat{t})\big)$. It can be easily checked that $\widehat{A_l}(x,t)$ is an infinitesimal of order $L^3$ and thus on the $\mathcal{ST}$**-lattice** we have,

$$\frac{\Delta\widehat{R}(\hat{x},\hat{t})}{\Delta t} \approx \frac{\widehat{J}(\hat{x},\hat{t}) - \widehat{J}(\hat{x}+L,\hat{t})}{L},. \tag{9}$$

Equation (9) on standard real time and space lines becomes $\partial_t R(x,t) = -\partial_x J(x,t)$. As a consequence of (8) and (9), even if we start with the initial condition (5), after some time $\hat{t} > 0$ we would have,

$$\sum_{\hat{x}\in[\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} L\,\widehat{R}(\hat{x},\hat{t}) \quad \approx \quad 1. \tag{10}$$

If we want to interpret $\widehat{R}(\hat{x},\hat{t})$ as a density of position states on the $\mathcal{ST}$**-lattice** we have to have equal sign $(=)$ instead of $\approx$ in (10), in which case $\widehat{R}(\hat{x},\hat{t})$ is not appropriate. We resolve this problem by making a subtle point that we have an *ontological wave function*,

$$\widehat{\mathbf{\Psi}}_{ont}(\hat{x},\hat{t}) = \sqrt{\widehat{N}(\hat{x},\hat{t})}\exp(i\,\widehat{\theta}(\hat{x},\hat{t})), \tag{11}$$

where $\widehat{N}(\hat{x},\hat{t})$ (infinitely large) is the number of *positions states* at $(\hat{x},\hat{t})$. We may assume $\widehat{\mathbf{\Psi}}_{ont}$ evolves similarly to (6) as,

$$\begin{aligned}\widehat{\mathbf{\Psi}}_{ont}(\hat{x},\hat{t}+\Delta t) \quad &= \quad \widehat{\mathbf{\Psi}}_{ont}(\hat{x},\hat{t}) - i\,\widehat{\mathbf{\Psi}}_{ont}(\hat{x},\hat{t}) + \frac{i}{2}\big(\widehat{\mathbf{\Psi}}_{ont}(\hat{x}-L,\hat{t}) + \widehat{\mathbf{\Psi}}_{ont}(\hat{x}+L,\hat{t})\big) \\ &\quad -\frac{i}{\hbar}\widehat{v}(\hat{x})\widehat{\mathbf{\Psi}}_{ont}(\hat{x},\hat{t})\Delta t. \end{aligned} \tag{12}$$

We define the *density of position states*, $\mathbf{R}(\hat{x},\hat{t})$ as,

$$\widehat{\mathbf{R}}(\hat{x},\hat{t}) = \frac{\widehat{N}(\hat{x},\hat{t})}{L\widehat{N}_{Total}(\hat{t})}, \tag{13}$$

where $\widehat{N}_{Toltal}(\hat{t})$ is the total number of position states,

$$\widehat{N}_{Toltal}(\hat{t}) = \sum_{\hat{y}\in[\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} \widehat{N}(\hat{y},\hat{t}),$$

and thus by definition we have,

$$\sum_{\hat{x}\in[\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} \widehat{\mathbf{R}}(\hat{x},\hat{t})\, L \;=\; 1.$$

In equation (6) note that if we take $\widehat{\zeta}(\hat{x}) = \sqrt{\frac{\widehat{N}(\hat{x},0)}{L\widehat{N}_{Total}(0)}}\exp(i\,\widehat{\theta}(\hat{x},0)) = \frac{\widehat{\mathbf{\Psi}}_{ont}(\hat{x},0)}{\sqrt{L\widehat{N}_{Total}(0)}}$, by comparing (6) and (12) (which are similar) at time $\hat{t}>0$ we have $\widehat{\psi}(\hat{x},\hat{t}) = \sqrt{\frac{\widehat{N}(\hat{x},t)}{L\widehat{N}_{Total}(0)}}$ $\exp(i\,\widehat{\theta}(\hat{x},\hat{t}))$ and then from comparing this and (10) we find,

$$\widehat{N}_{Toltal}(\hat{t})/\widehat{N}_{Toltal}(0) \approx 1,$$

and therefore,

$$\widehat{R}(\hat{x},\hat{t}) = \frac{\widehat{N}(\hat{x},\hat{t})}{L\widehat{N}_{Total}(0)} \;\approx\; \frac{\widehat{N}(\hat{x},\hat{t})}{L\widehat{N}_{Total}(t)} = \widehat{\mathbf{R}}(\hat{x},\hat{t}).$$

In other words $\widehat{\mathbf{R}}(\hat{x},\hat{t})$ is just the normalization of, $\widehat{R}(\hat{x},\hat{t})$, given by,

$$\widehat{\mathbf{R}}(\hat{x},\hat{t}) = \frac{\widehat{R}(\hat{x},\hat{t})}{\sum_{\hat{x}\in[\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} L\,\widehat{R}(\hat{x},\hat{t})} \;\approx\; \widehat{R}(\hat{x},\hat{t}).$$

As we argue in section 2.3 the quantity on the $\mathcal{ST}$-**lattice** that appears in statistical expressions and corresponds to probability density on standard real time and space axes is, $\widehat{\mathbf{R}}(\hat{x},\hat{t})$. Thus the true analogue of wave function $\psi(x,t) = \sqrt{R(x,t)}\exp(i\,\theta(x,t))$ on the $\mathcal{ST}$-**lattice** is the statistical image of $\widehat{\mathbf{\Psi}}_{ont}(\hat{x},\hat{t})$ defined as,

$$\widehat{\Psi}_{stat}(\hat{x},\hat{t}) = \sqrt{\widehat{\mathbf{R}}(\hat{x},\hat{t})}\exp(i\,\widehat{\theta}(\hat{x},\hat{t})) = \sqrt{\frac{\widehat{N}(\hat{x},\hat{t})}{L\widehat{N}_{Total}(\hat{t})}}\exp(i\,\widehat{\theta}(\hat{x},\hat{t})). \tag{14}$$

For mathematical precision we need to mention some points about equation (12). First we note that, for every point, $\hat{x}$, at a fixed time, $\hat{t}$, in the hyperfinite lattice we should have either $\widehat{N}(\hat{x},\hat{t}) = 0$ or $\widehat{N}(\hat{x},\hat{t}) \in {}^*\mathbb{N}\setminus\mathbb{N}$, otherwise by (12) we may encounter a rational or infinitesimal number of position states at a later time, $\hat{t}' > \hat{t}$ for some spatial points. Even if we impose some conditions on initial ontological wave function such that for every $\hat{x}$ in the hyperfinite lattice we have either $\widehat{N}(\hat{x},0) = 0$ or $\widehat{N}(\hat{x},0) \in {}^*\mathbb{N}\setminus\mathbb{N}$ large enough, so that at any later time ($\hat{t}>0$), $\widehat{N}(\hat{x},\hat{t})$ is either zero or infinite, it is not evident from (12) that after $t=0$ whether $\widehat{N}(\hat{x},\hat{t})$ (if it is infinite) remains infinite hyperinteger number or becomes an infinite hyperreal number. For the PBM model, equation (14) and the meaning of $\widehat{N}(\hat{x},\hat{t})$ as the number of position states on which particles jump after each time $\Delta t$ (as will be explained in the next section) is fundamental. We can state some conditions weaker than equation (12) for the mathematical existence of ontological wave function. Let us express the following *existence statement* (proposition), for the existence of ontological wave function (and subsequently the statistical wave function) on $\mathcal{ST}$-**lattice** required for the PBM model,

**Proposition 1.** *Suppose $\widehat{\psi}(\hat{x}, \hat{t})$ is defined by initial condition $\zeta$ (with (5) valid) and time evolution equation (6) on the $\mathcal{ST}$-$\mathtt{lattice}$, then there exists an ontological wave function $\widehat{\Psi}_{ont}(\hat{x}, \hat{t})$ given by (11) and (consequently) statistical wave function $\widehat{\Psi}_{stat}(\hat{x}, \hat{t})$ given by (14) such that for all $\hat{x}$ and $\hat{t}$ on the $\mathcal{ST}$-$\mathtt{lattice}$, either $\widehat{N}(\hat{x}, \hat{t}) = 0$ or $\widehat{N}(\hat{x}, \hat{t}) \in {}^*\mathbb{N} \setminus \mathbb{N}$ and also we have,*

$$\widehat{\Psi}_{stat}(\hat{x}, \hat{t}) = \widehat{\psi}(\hat{x}, \hat{t}) + \mathcal{O}((\Delta t)^{\alpha})$$

*where $\alpha > 1$ ($\alpha$ is real number) and $\mathcal{O}((\Delta t)^{\alpha})$ is an infinitesimal term of order $(\Delta t)^{\alpha}$ (e.g. for any bounded values of $\hat{x}$ and $\hat{t}$ on the $\mathcal{ST}$-$\mathtt{lattice}$ there exists a bounded number $k$, such that $|\mathcal{O}(\Delta t)^{\alpha}| < k(\Delta t)^{\alpha}$).*

The above conditions are weaker than equation (12) since, in this case, $\widehat{N}_{Total}(\hat{t})$ and $\widehat{N}_{Total}(\hat{t} + \Delta t)$ may not be related at all. This can be visualized intuitively and easily proven. Finally we should note that if $\widehat{\psi}(\hat{x}, \hat{t})$ is a function which is described in the above statement then any $\widehat{\psi}'(\hat{x}, \hat{t}) = \widehat{\psi}(\hat{x}, \hat{t}) + \mathcal{O}((\Delta t)^{\alpha})$, (where $\alpha > 1$) on real space and time lines is also valid in the Schrödinger equation and has a same image. Thus we can say such statistical wave functions correspond to the same class of ontological wave functions.

### *2.3. Describing particle motion and wave function collapse as probabilistic particle jumps*

In this section again, for simplicity, we present arguments using the one particle Schrödinger equation in one dimension, but we generalize to the case of the 3 dimensional $n$-particle Schrödinger equation in section 2.4. Any path a particle can take on standard real axes $(x, t)$, can be imagined as being produced by the assumption of the particle's (center of mass) jump from one point $(\hat{x}, \hat{t})$, to the neighboring points $(\hat{x} \pm L, \hat{t} + \Delta t)$ on the $\mathcal{ST}$-$\mathtt{lattice}$, since the speed for such a jump is infinite, $V_{\mathrm{J}ump} = L/\Delta t = \hbar/mL \sim \infty$. Clearly, infinite speed of a particle on the $\mathcal{ST}$-$\mathtt{lattice}$ does not necessarily mean infinite speed on standard real lines, as the particle can be imagined as moving forward and backward on the $\mathcal{ST}$-$\mathtt{lattice}$ while remaining still on the standard real axis for the finite standard real time interval. The exact way particles move on the $\mathcal{ST}$-$\mathtt{lattice}$ is hidden from standard real space and time lines and we have no choice but to look for the models that lead to the same statistical results as $QM$ on standard real time and space lines.

One natural choice to model particle jump is to assume a particle at location $\hat{x}$ at time $\hat{t}$ will jump randomly, after time $\Delta t$, according to position states on each side $\widehat{N}(\hat{x} \pm L, \hat{t} +$

$\Delta t$), to one of neighboring points, $\hat{x} \pm L$,

$$(\hat{x}, \hat{t}) \;\rightarrow\; (\hat{x} \pm L, \hat{t} + \Delta t).$$

So, if the particle is at $(\hat{x}, \hat{t})$ the probability for a jump to the right side after $\Delta t$, $\mathcal{P}_A\big((\hat{x}, \hat{t}) \rightarrow (\hat{x} + L, \hat{t} + \Delta t)\big)$ and the probability for a jump to the left side, $\mathcal{P}_A\big((\hat{x}, \hat{t}) \rightarrow (\hat{x} - L, \hat{t} + \Delta t)\big)$ are given by,

$$\mathcal{P}_A\Big((\hat{x}, \hat{t}) \rightarrow (\hat{x} \pm L, \hat{t} + \Delta t)\Big) = \frac{\widehat{\mathbf{R}}(\hat{x} \pm L, \hat{t} + \Delta t)}{\widehat{\mathbf{R}}(\hat{x} + L, \hat{t} + \Delta t) + \widehat{\mathbf{R}}(\hat{x} - L, \hat{t} + \Delta t)}. \tag{15}$$

The above probabilities guide the particle to positions where $\widehat{\mathbf{R}}$ has greater values. The particle is moving with infinite speed, randomly passing all spatial points within the wave function, where $\widehat{\mathbf{R}}(\hat{x}, \hat{t}) \neq 0$, at any instant of standard real time, $t = \mathbf{st}(\hat{t})$. The particle motion according to (15) is consistent with Born's statistical rule since, the quantity on the $\mathcal{ST}$-**lattice**, which statistically appears on standard real space and time lines, is $\widehat{\mathbf{R}}(\hat{x}, \hat{t})$, which is the probability of finding the particle at $(\hat{x}, \hat{t})$. Thus, the mean location of the particle is found by,

$$\overline{x_t} = \sum_{\hat{x} \in [\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} \hat{x}\, \widehat{\mathbf{R}}(\hat{x}, \hat{t})\, L \;\approx\; \int_{-\infty}^{+\infty} dx\, x\, R(x, t) = \int_{-\infty}^{+\infty} dx\, \psi^*(x, t)\, x\, \psi(x, t),$$

and the average velocity of such distribution is found by adding the current density of all components,

$$\begin{aligned}
\overline{v_t} \;&\approx\; \sum_{\hat{x} \in [\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} \widehat{J}(\hat{x}) L \\
&\approx\; \frac{\hbar}{2im}\Big( \int_{-\infty}^{+\infty} dx\, \psi^*(x, t)\, \partial_x\, \psi(x, t) - \int_{-\infty}^{+\infty} dx\, \psi(x, t)\, \partial_x\, \psi^*(x, t) \Big) \\
&=\; \frac{\hbar}{i\, m} \int_{-\infty}^{+\infty} dx\, \psi^*(x, t)\, \partial_x\, \psi(x, t).
\end{aligned}$$

The average momentum is $\overline{P_t} = m\overline{v_t} \approx \int_{-\infty}^{+\infty} dx\, \psi^*(x, t)\, \frac{\hbar}{i} \partial_x\, \psi(x, t)$, thus the $QM$ position and momentum operators ($X : \psi(x) \rightarrow x\psi(x)$ and $P : \psi(x) \rightarrow -i\hbar\partial_x\psi(x)$), appear naturally in the expectation values on standard real time and space lines. The definition of the inner product as well as one particle position and momentum states on the $\mathcal{ST}$-**lattice** is brought in appendix A.

Clearly, in the PBM model the particles do not have paths on standard real space and time axes (even in the relativistic domain, as will be discussed in section 3). This resembles, *the only wave view interpretations of quantum mechanics*, as there is no point on the real line where at any instant of real standard time the particle's center of mass can be assumed to remain still. These considerations raise the issue of particle observation, if the particles do not have paths on real time and space lines how do we observe them? We should remember that in the PBM model we interpreted the probability density, as the density of position states on hyperreal axes, which makes wave function change (collapse) inevitable, because the particle, moving with infinite speed, is present in all split parts of wave function (e.g. in a Stern Gerlach experiment) and only the wave function collapse in a measurement can change the future measurement outcomes as predicted by quantum mechanics. The Born rule states that if an observable, corresponding to a self-adjoint operator $A$ with discrete spectrum ($\lambda_1$,$\lambda_2$,...), is measured in a system with normalized wave function $\psi$ then; the measured result will be one of the eigenvalues $\lambda$ of $A$ and the probability of measuring a given eigenvalue $\lambda_i$ will equal $\langle\psi|u_i\rangle\langle u_i|\psi\rangle = |\langle\psi|u_i\rangle|^2$ (where $u_i$ is the eigenvector corresponding to eigenvalue $\lambda_i$).

In the following we discuss the Born rule and wave function collapse in more detail, by considering the above rules for two different observations on the $\mathcal{ST}$-$\mathtt{lattice}$. Firstly the case where observation is related to an observable with discrete and countable set of eigenvectors on real axes and secondly, observation of particle location in which case the eigenvectors on real axes are uncountably infinite. Now lets consider the first case, where the observable $A$ has a discrete and countable set of eigenvectors $\{u_1, u_2, ...\}$ corresponding to a set of eigenvalues $\{\lambda_1, \lambda_2, ...\}$. The wave function $|\psi\rangle$ can be equivalently written in the base $\{u_1, u_2, ...\}$ as,

$$\psi(x) = \sum_i \alpha_i u_i(x),$$

where $\alpha_i = \langle u_i|\psi\rangle$. Thus we have,

$$\psi(x)\psi^*(x) = \sum_i |\alpha_i|^2 u_i(x)\, u_i^*(x) + \sum_{i>j} (\alpha_i \alpha_j^*\, u_i(x)\, u_j^*(x) + cc). \tag{16}$$

The wave function collapse to one of eigenvectors $u_1, u_2, ...$ has two steps. First, the position density of states (probability density) undergoes changes and the second term in (16) (the cross terms) disappears due to decoherence (e.g. see [9] and [26]),

$$|\psi(x)|^2 \to \sum_i |\alpha_i|^2 |u_i(x)|^2.$$

The decoherence takes place after a finite time, known as decoherence time, therefore the process happens on standard real time and space axes and for this reason the above equations were written using standard real axes. Although the second term in (16) when summed up on the whole space line is zero (since the set of eigenvectors are orthogonal), except in the case of wave collapse, their contributions are always present. It is as if, before the collapse, the contribution of wave function components ($\alpha_i u_i(x)$ for $i = 1, 2, 3, \cdots$)

to the density of position states, as supposed to be produced by their linear superposition, becomes independent. The second and final step in collapse process is the particle jump. After decoherence, the particle at location $\hat{x}$ (at time $\hat{t}$) jumps to one of the classes of position states of neighboring sites $|\alpha_i|^2|\widehat{u}_i(\hat{x} \pm L)|^2$ for $i = 1, 2, ...$ after time $\Delta t$, which will finish the collapse process. All other classes of position states become irrelevant, thus the density of position states changes from $|\alpha_i|^2|\widehat{u}_i(\hat{x})|^2$ to the the unit vector $|\widehat{u}_i(\hat{x})|^2$ immediately. From the assumptions of the PBM model the best way to model the particle jump subsequent to decoherence is as following. *The probability of getting a $\lambda_i$ result in the measurement over an ensemble, when the wave function $\widehat{\psi}$ collapse to $\widehat{u}_i(\hat{x})$ is finalized by particle jumping from neighboring sites to the point $\hat{x}$, is given by*

$$P(\lambda_i|\hat{x}) = |\alpha_i|^2|\widehat{u}_i(\hat{x})|^2 L. \tag{17}$$

Then the probability of getting $\lambda_i$ result in the measurement over the ensemble is derived from the above as,

$$P(\lambda_i) = \sum_{\hat{x}} P(\lambda_i|\hat{x}) = \sum_{\hat{x}} |\alpha_i|^2|\widehat{u}_i(\hat{x})|^2 L = |\alpha_i|^2. \tag{18}$$

The equation (17) is essential in the PBM model to ensure that particle trajectory is continuously consistent with density of position states, such that the particle goes to the points where position states exist ($\widehat{u}_i(\hat{x}) \neq 0$) and are more abundant.

Now lets consider the second case, the observation of the location of a particle (with wave function $\psi$) on the $\mathcal{ST}$-**lattice**. In other words hypothetical detectors are placed at every point on the hyperreal line to detect the location of the particle. The sum of probabilities for detection at each point over the whole line, according to the Born rule, is given by

$$\sum_{\hat{x}} \langle \widehat{\psi}\ |\widetilde{\hat{x}}\rangle\langle\widetilde{\hat{x}}|\ \widehat{\psi}\rangle = \sum_{\hat{x}} \widehat{\psi}(\hat{x})\, \widehat{\psi}^*(\hat{x})\, L = 1,$$

where normalized one particle position state $\widetilde{v}_{\hat{x}_1}(\hat{x}) \equiv |\widetilde{\hat{x}_1}\rangle$ is defined in relations (A.2). On standard real axes, this can be done by choosing a measure (for integration) to write the above line in integral form as, $\int_{-\infty}^{\infty} |\langle\psi|x\rangle\langle x|\psi\rangle| dx = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$. For this case (position detection) we can also examine the formula (17). *The probability of observing the particle at point $\hat{x}_1$ due to the collapse of the wave function which is finalized by the particle jump from neighboring sites to point $\hat{x}$, is given by*

$$P(\hat{x}_1|\hat{x}) = |\widetilde{\langle\hat{x}_1}|\psi\rangle|^2|\widetilde{v}_{\hat{x}_1}(\hat{x})|^2\, L = |\widehat{\psi}(\hat{x}_1)|^2\delta_1(\hat{x} - \hat{x}_1)\, L^2,$$

where $\delta_1(\hat{x} - \hat{x}_1)$ (the delta Dirac function) is defined in relations (A.1). As expected, the probability of getting the value $\hat{x}_1$, in the position measurement, is derived as

$$P(\hat{x}_1) = \sum_{\hat{x}} P(\hat{x}_1|\hat{x}) = \sum_{\hat{x}} |\widehat{\psi}(\hat{x}_1)|^2\delta_1(\hat{x} - \hat{x}_1)\, L^2 = |\widehat{\psi}(\hat{x}_1)|^2\, L.$$

In the PBM model the particle traces in trace chambers can be attributed to continual wave function collapses (localization). The distance between each of two subsequent collapses is approximately equal to the group velocity of the wave packet multiplied by the decoherence time.

For the PBM model the question which remains is, at which level of decoherence the collapse happens and how. In the existing collapse theories (e.g. [18], [16] and [23]) the Schrödinger equation is supplemented with additional nonlinear and stochastic terms (with phenomenological parameters) which localizes the wave function in space. For systems consisting of many particles the stochastic collapse dynamics become stronger than the quantum dynamics via the amplification mechanism. For the PBM model we have hidden variables (the location of particles on the $\mathcal{ST}$-**lattice**) which can be used to model the collapse, for example as a result of some kind of collision (as we did above in the description of observing the location of a particle) in which case the collapse will be stronger when the system involves more particles.

*2.4. Multiparticle non-relativistic $QM$ and identical particles* At the end of this section

we generalize our one particle non-relativistic one dimensional wave equation on the $\mathcal{ST}$-**lattice** to the $n$-particle three dimensional wave equation. Let us consider the following wave equation on standard real axes,

$$\sum_{j=1}^{n}\frac{-\hbar^2}{2m_j}\nabla_j\,\psi'(\vec{\mathbf{x}},t)+\sum_{r>j}\widehat{v}(|\vec{x_r}-\vec{x}_j|)\psi'(\vec{\mathbf{x}},t)=i\hbar\partial_t\psi'(\vec{\mathbf{x}},t), \tag{19}$$

where $\vec{\mathbf{x}}=(\vec{x}_1,\vec{x}_2,...,\vec{x}_n)$ is a vector in configuration space. We introduced configuration space and time of $n$ particles with masses $m_1, m_2$,...$m_n$ on the $\mathcal{ST}$-**lattice** by (3). Taking $\widehat{\mathbf{x}}=(\widehat{x}_1,\widehat{x}_2,...,\widehat{x}_n)$, where $\widehat{x}_i=(\widehat{x}_i^1,\widehat{x}_i^2,\widehat{x}_i^3)$ and $\widehat{\mathbf{L}}_i^r=(0,\ldots,0,\widehat{L}_i^r,0,\ldots 0)$ (at $i-th$ place), $r=1,2,3$ where $\widehat{L}_i^1=(L_i,0,0)$, $\widehat{L}_i^2=(0,L_i,0)$ and $\widehat{L}_i^3=(0,0,L_i)$ (the infinitesimal lengths, $L_i$'s, were defined by (2)), in configuration space, the recursive time evolution equation for the wave function $\widehat{\psi}'(\widehat{\mathbf{x}},\hat{t})$ on the $\mathcal{ST}$-**lattice** can be written as,

$$\begin{aligned}\widehat{\psi'}(\widehat{\mathbf{x}},\hat{t}+\Delta t) &= \widehat{\psi'}(\widehat{\mathbf{x}},\hat{t})-3\,i\,\widehat{\psi'}(\widehat{\mathbf{x}},\hat{t})+\frac{i}{2}\sum_{i=1}^{n}\sum_{r=1}^{3}\left(\widehat{\psi'}(\widehat{\mathbf{x}}-\widehat{\mathbf{L}}_i^r,\hat{t})+\widehat{\psi'}(\widehat{\mathbf{x}}+\widehat{\mathbf{L}}_i^r,\hat{t})\right)\\ &\quad -\frac{i}{\hbar}\sum_{i>j}\widehat{v}(|\widehat{x}_i-\widehat{x}_j|)\,\widehat{\psi'}(\widehat{\mathbf{x}},\hat{t})\Delta t.\end{aligned} \tag{20}$$

Since, for points $\widehat{x}_i$ and $\widehat{x}_j$, the units of measurement $L_i$ and $L_j$ might be different, we have to measure $\widehat{x}_i-\widehat{x}_j$ with unit $L_{i,j}$ where $L_{i,j}=\max\{L_i,Lj\}$. This is the lowest value by which the distance between two masses $m_i$ and $m_j$ can be measured. It can be easily shown that equation (20) on standard real space and time lines leads to (19). The jump of

$n$ particles in this case can be characterized by a single jump in configuration space, from one point $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_n)$ , after time $\Delta t$, to one of its neighboring points $\widehat{\mathbf{x}'} \in S(\widehat{\mathbf{x}})$,

$$S(\widehat{\mathbf{x}}) = \{\widehat{\mathbf{x}'} \mid \widehat{\mathbf{x}'} = (\widehat{x}_1 \pm \widehat{L_1^{r_1}}, \widehat{x_2} \pm \widehat{L_2^{r_2}}, \ldots, \widehat{x}_n \pm \widehat{L_n^{r_n}})\}, \qquad r_i = 1, 2, 3 \text{ and } i = 1, 2, ..., n.$$

The set of neighboring points, $S(\widehat{\mathbf{x}})$, has a total number of $6^n$ points. The jumping probability, analogous to (15), in this case, is given by

$$\mathcal{P}_A\Big((\widehat{\mathbf{x}}, \hat{t}) \rightarrow (\widehat{\mathbf{y}}, \widehat{t} + \Delta t)\Big) = \frac{\widehat{R}'(\widehat{\mathbf{y}}, \widehat{t} + \Delta t)}{\sum_{\widehat{\mathbf{x}'} \in S(\hat{\mathbf{x}})} \widehat{R'}(\widehat{\mathbf{x}'}, \hat{t} + \Delta t)}, \qquad \widehat{\mathbf{y}} \in S(\widehat{\mathbf{x}}). \tag{21}$$

The inclusion of spin in the waves equation is straightforward and can be done, but it is ignored it in section.

*Identical particles*

The statement about the non-relativistic wave function of $n$ identical particles, $\psi(\vec{x}_1, \vec{x}_2, ..\vec{x}_n)$, is that, the dynamics of the system at any given moment of time is independent of any permutation of particle positions. This condition in Born statistical interpretation of $QM$ is written as,

$$|\psi(\vec{x}_1, \vec{x}_2, ..\vec{x}_n)|^2 = |\psi(\pi(\vec{x}_1, \vec{x}_2, ..\vec{x}_n))|^2 \tag{22}$$

In Bohmian mechanics if (22) is satisfied, the guiding wave equations also satisfy the above condition about the dynamic of the system. The same is true in the PBM model in which the jump probability (21) is independent of any permutation of particle positions, if (22) is satisfied. The result of (22) is that the wave function is either symmetric or antisymmetric depending on whether or not, two (or more) particles can be in the same position at any given moment of time.

## 3. Relativistic Extension

In this section we want to discuss the relativistic extension of the non-relativistic PBM model, we presented in the previous section. In the extension of Bohmian mechanics to special relativity, the need for preferred foliation (to define simultaneity) arises for the case of guiding equations of a multi-particle system, since the Bohm velocity for each particle not only depends on the wave function but also on locations of all other particles at the same time. In the PBM model we have the same need for defining simultaneity because the non-local nature of motions of $n$ particles, described as a single jump in configuration space, is evident from the probability formula (21), which depends on the locations of all particles at a given time. For simplicity, in this paper, we assume the flat foliation of Minkowski space is introduced by a *preferred Lorentz frame of reference* where the simultaneity is defined.

Thus, the next step would be to define the $\mathcal{ST}$-**lattice**. Once defined, the Dirac wave equation of single particle,

$$\beta m C^2 \psi(\vec{x},t) - i\hbar C \sum_{n=1}^{3} \alpha_n \partial_{x_n} \psi(\vec{x},t) = i\hbar \partial_t \psi(\vec{x},t),$$

can be made discrete using infinitesimals $L \approx 0$ and $\Delta t \approx 0$ as,

$$\widehat{\psi}(\widehat{\mathbf{x}},\hat{t}+\Delta t)-\widehat{\psi}(\widehat{\mathbf{x}},\hat{t}) = -i\frac{\beta m C^2 \Delta t}{\hbar}\psi(\widehat{\mathbf{x}},\hat{t}) - \frac{C\Delta t}{2L}\sum_{r=1}^{3}\alpha_r(\widehat{\psi}(\widehat{\mathbf{x}}+\widehat{L^r},\hat{t})-\widehat{\psi}(\widehat{\mathbf{x}}-\widehat{L^r},\hat{t})), \tag{23}$$

where $\widehat{\mathbf{x}} = (\hat{x}_1,\hat{x}_2,\hat{x}_3)$, $\widehat{L}^1 = (L,0,0)$, $\widehat{L^2} = (0,L,0)$ and $\widehat{L^3} = (0,0,L)$. The jump probability (analogous to (21)) for a single particle, described by the Dirac equation, is given by

$$\mathcal{P}_A\Big((\widehat{\mathbf{x}},\hat{t}) \to (\widehat{\mathbf{x}} \pm \widehat{L}^i, \hat{t}+\Delta t)\Big) = \frac{\widehat{\psi}^\dagger\ \widehat{\psi}|_{(\widehat{\mathbf{x}}\pm\widehat{L}^i,\hat{t}+\Delta t)}}{\sum_{i=1}^{3}\big(\widehat{\psi}^\dagger\ \widehat{\psi}|_{(\widehat{\mathbf{x}}+\widehat{L}^i,\hat{t}+\Delta t)} + \widehat{\psi}^\dagger\ \widehat{\psi}|_{(\widehat{\mathbf{x}}-\widehat{L}^i,\hat{t}+\Delta t)}\big)}. \tag{24}$$

The main question which remains here is, the relation between $L$ and $\Delta t$, the answer to which lies in the way we assume the particle moves .

One approach for this is to use the uncertainty principle, as done in the previous section, applying relativistic relations (e.g. $m({L_R}^2/\Delta\tau)/\sqrt{(1-(L_R/\Delta\tau C)^2)} \geq \hbar$) to define the relationship between $L_R$, $\Delta t_R$ (the maximum of $\Delta\tau$) and $m$, (which is $(\Delta t_R)^2 = L_R^2/C^2 + m^2\, L_R^4/\hbar^2$). The result of this view is that the particle moves with speed $C$ minus an infinitesimal value ($V_{Jump} \approx C$). Thus, on real standard time and space axes, particles move with speed $C$ and therefore have paths on real space and time lines (similar to $\mathcal{BM}$). However, as we discussed before, because of finite speeds for particles we need the assumption of the quantum equilibrium hypothesis in order to be consistent with Born's statistical rule. Thus, with this approach, we may end up with another version of Bohmian mechanics which should have the quantum equilibrium hypothesis as a principle.

The second approach to define the space and time hyperfinite dimensional lattice, which, in this section, we consider as the appropriate way to extend PBM to the relativistic domain, is to assume that particles move with infinite speed in the preferred frame of reference. Although this seems to contradict special relativity, what we argue here is that: all the relativistic effects we perceive in our devices when measuring time, length, energy etc., is due only to consistency of the wave equations with special relativity on real standard space and time lines and not due to consistency of particle motions with special relativity. To define $\mathcal{PST}$-**lattice** for the preferred frame, according to what is explained, it is sufficient to choose a relationship (between $L$ and $\Delta t$) which indicates infinite speed for a particle. The relationship (2) we used in the non-relativistic case has this property. Thus by inserting,

$$L \approx 0, \qquad\qquad (\Delta t) = \frac{m\,L^2}{\hbar},$$

into (23) we derive,

$$\widehat{\psi}(\widehat{\mathbf{x}}, t\hat{+}\Delta t) - \widehat{\psi}(\widehat{\mathbf{x}}, \hat{t}) = -i\frac{\beta\,m^2\,C^2 L^2}{\hbar^2}\widehat{\psi}(\widehat{\mathbf{x}}, \hat{t}) - \frac{m\,C\,L}{2\,\hbar}\sum_{r=1}^{3}\alpha_r(\widehat{\psi}(\widehat{\mathbf{x}}+\widehat{\mathbf{L}}^r, \hat{t}) - \widehat{\psi}(\widehat{\mathbf{x}}-\widehat{\mathbf{L}}^r, \hat{t})). \tag{25}$$

By assuming infinite speed for particles, although they have paths on the space time hyperfinite dimensional lattice of the preferred Lorentz frame, they may not have paths on all standard real time and space lines of Lorentz frames. The space and time hyperfinite dimensional lattice of the preferred Lorentz frame, $\mathcal{PST}$-**lattice**, is unique in the sense that, it can only be defined for the preferred Lorentz frame and where space and time position states are defined. All Lorentz frames at each instant of real standard time perceive the presence of the particle on all of those position states that are observable, which is the Lorentz transformation of wave function in the preferred frame of reference. The tail speed of wave function $\widehat{\psi}$ on real standard time and space lines of the preferred Lorentz frame (and any other Lorentz frame) is equal to $C$, although on the $\mathcal{PST}$-**lattice** some infinitesimal values spread with infinite speed $\hbar/mL$. The jumping probability for a single electron is given by (24) where $L$ and $\Delta t$ are replaced by (2). The PBM model, as an *ontological wave collapse theory*, has a straightforward solution for *the causality problem* (related to the wave function collapse) given by preferred foliation. For any space like entangled measurement there is always a sequence of events (cause and effect) which is determined by preferred foliation. The well known example is the measurement of spins of two entangled particles, with zero total spin, by two space like apart observations (observers). Different Lorentz frames disagree on which observer (in the spin measurements) has done the observation first (consequently causing wave function collapse). In the PBM model this issue can only be resolved in the preferred frame of reference (preferred space-time foliation). Although, similar to the preferred space-time foliation itself, the sequence of such events is unobservable.

For the motion of photons according to relation (2) one should assume an infinitesimal rest mass for photons, in which case the ratio of infinitesimal length of photons motion to infinitesimal length of motion of any other particle with real non zero mass would be infinite. For example assuming the relation $m_p = (C^2/G)L_p$ (where $G$ is gravitational constant) between mass of photon and its infinitesimal length of motion, then by (2) we get $L_p = ((G\hbar/C^2)\Delta t)^{\frac{1}{3}}$. In this case it seems the appropriate choice for field equations (for motion of photons) would be Proca field equations with infinitesimal mass (which then should be expressed on the $\mathcal{PST}$-**lattice**). As we know, relativistic wave mechanics (e.g. Dirac equation) can not account for the problems of negative energy presence and also particle-anti particle creation and annihilation. For example, a one particle wave function, which is initially localized in the bonded region of space, is a combination of positive and negative energy components. Even if we start from a wave packet composed of positive energies, the negative energy components will appear, leading to so called Zitterbewgung oscillations. Thus, the positive energy states are always accompanied by negative energy states. These issues are resolved in $QFT$ where for example Zitterbewgung oscillations are attributed to spontaneously forming and annihilating electron-positron pairs. In $QFT$ one uses the extended modes (plane waves), which are solutions of free wave equations. In the PBM model the elements of Fock space (plane waves) can well be described and conceptually attributed to particles as further explained in Appendix A. One of the important features of $QFT$ is locality; the statistics for measurement in one space-time region do not depend on whether or not a measurement has been performed in a space like related second space time region. In PBM extension to relativistic wave mechanics, as we saw, the wave function tail speed on standard real axes is $C$, which shows its consistency with the locality condition. We have brought an extra part in Appendix B on discussion about particle and field ontologies in $QFT$ and the PBM model .

## 4. Discussion

In this paper, we constructed an ontological interpretation of quantum mechanics which takes both particles and wave functions as objective realities. It gives the two objects (wave function and particles) simple and related meaning. One object, wave function, carrying information about density of position states which propagates like a wave in configuration space and the other, the particles, is what occupy these position states. Here some similarities and differences of the PBM model with some other interpretations of $QM$ such as Bohmian mechanics, objective collapse theories, many-worlds interpretation and Copenhagen interpretation, will be discussed.

*Bohm interpretation versus PBM*

PBM and $\mathcal{BM}$ are both hidden variable theories which have a lots of similarities in their theoretical construction, but in the end PBM is a kind of spontaneous collapse theory while $\mathcal{BM}$ does not need collapse postulate for the solution to measurement problem (although it leaves the issue of empty waves). Therefore despite their names PBM and $\mathcal{BM}$ belong to two different categories of Quantum interpretations. As Bell has shown (e.g. see [5]), any hidden variable interpretation of $QM$ is non-local, which means infinite speed (correlations) is embedded in hidden variable theories whether or not particles have trajectories consistent with newtonian mechanics (and special relativity for high energies). In $\mathcal{BM}$ we have trajectories on real space and time lines, at the cost of extra assumption (Quantum equilibrium hypothesis) and thus there is no need for collapse postulate. In the PBM model the assumption of infinite speed for particles on the hyperfinite dimensional lattice and jumping probability relations are replaced with quantum equilibrium hypothesis and guiding equations of $\mathcal{BM}$. With the assumption of infinite speed, the distinction between spatial extension of wave and particle traces at each instant of real time, is lifted, which necessarily makes the PBM model, a wave function collapse theory. Some may consider the fact that $\mathcal{BM}$ is free of collapse postulate as its advantage over objective collapse theories but apart from this and from the theoretical and ontological points of view, as a hidden variable theory, PBM has the advantage of having a simple and purely probabilistic ontology compared to $\mathcal{BM}$ ontology which has a more complicated structure (a combination of both probabilistic and deterministic processes).

*Objective collapse theories and PBM*

One of the ideas which has been formulated in response to explain how the definite outcomes appear after quantum measurement instead of their superposition as predicted by the $QM$, in another word how the classical world emerges from $QM$, is to recognize wave functions as every thing that is needed to interprets the quantum theory via spontaneous collapses. The existing models of this type, GRW theory, [18], as well as all subsequent developments (e.g. [16] and [23]) are phenomenological attempts to model the collapse process. The main problem in such collapse theories is whether they can be made compatible with relativistic requirements or not. The existing models are not compatible with relativity and although some attempts have been made, the final answer to the question is still unknown. The difficulty is how to combine the nonlocal character of the collapse, which is necessary in order to make it compatible with the experimentally verified violation of Bell inequalities, with the relativistic principle of locality. Further investigations, [15] and [17], have shown the formal structure of these theories is such that it does not allow, even conceptually, to establish cause-effect relations between space-like events. PBM is a new objective collapse theory. It has some features which do not exist in previous models. Firstly it incorporates particles with trajectories (on hyperfinite dimensional lattices and not on real space-time axes). The existence of hidden variables, the location of particles on the $\mathcal{ST}$-$\mathtt{lattice}$, in the PBM model, might be used to model instantaneous wave reduction. Secondly it should be noted that non-locality is embedded in the PBM model, as for example the jumping probabilities for particles (relations (21)) are non-local and also the model has a preferred space-time foliation. These considerations in the PBM model makes the non-local nature of collapses compatible with the locality principle of $QM$. The preferred space-time foliation ($\mathcal{PST}$-$\mathbf{lattice}$) can determine the sequence of collapses related to space like entangled systems and although this sequence is non-observable (similar to preferred foliation itself which can not be realized on real standard axes), the principle of causality is preserved.

*Many-worlds interpretation of $QM$ versus PBM*

PBM is in contrast with the many-worlds interpretation in two important aspects. First, unlike the many-worlds interpretation where wave function never collapses, in PBM one should assume wave function reduction. Secondly, unlike the many-worlds interpretation, the PBM model is a hidden variable theory, describing a single history. The advantage of Many world interpretation over $\mathcal{BM}$ and the existing collapse theories is its simplicity and its compatibility with relativity. The collapse postulate and world branching have a more or less equivalent role in their related interpretations. In this respect, PBM is even more similar to many world interpretation as it is able to model instantaneous localization (without any side effects) on real time and space lines. The PBM model also has a simple ontology which is close to conventional formalism of quantum mechanics.

*Copenhagen interpretation versus PBM*

Copenhagen interpretation has some interesting similarities with the PBM model, such that one can say PBM completes Copenhagen interpretation by resolving its main problem which is the subjectivity of observation. Copenhagen interpretation asserts that particles do not have positions and momentums before the measurement. This is also true in the PBM model since, before the wave function reduction, particles do not have specific locations on standard real space and time axes. However, in PBM, as a hidden variable theory, particles positions and trajectories exist in hyperfinite dimensional space and time lattice, hidden from our measuring devices (which work only on real-space time axes). Copenhagen interpretation has difficulties in defining exactly what is the observer and what is the system observed. Thus it encounters paradoxes like Schrödinger cat. The PBM model, however, as an objective collapse theory is free of such paradoxes. Despite paradoxes and lack of consensus about all aspects of the model, the Copenhagen interpretation is still one of the most popular quantum interpretations among physicists in regard to thinking about quantum phenomena [27]. Resolving its shortcomings within a consistent logical frame work (like PBM), even if it does not lead to any new result on standard real axes, is still valuable from a pedagogical point of view.

At the end the choice between different interpretations of $QM$ is a matter of taste unless some new experimental findings support for example $\mathcal{BM}$ or objective collapse theories, giving them advantages. The criterion on hand now is that the interpretation is complete and free of contradictions. One may criticize that the PBM model relies on hyperreal numbers, which are historically considered as ideal elements (e.g. abstract notions that are useful as tools but do not correspond to anything real in the outside world). For this we should remember that the quantum mechanics is at the center of the science of physics, to understand it the well-known many world interpretation assumes the existence of parallel universes, so it should not be strange to use methods based on the assumption of the existence of infinite and infinitesimal values (of speed, length, time and etc.) in nature, for this end.

**Appendices**

*A. Position and momentum states*

Here we consider defining one particle position and momentums states and their orthogonality conditions on $\mathcal{ST}$-$\mathtt{lattice}$. Since, in the PBM model, $\widehat{\mathbf{R}}(\hat{x},\hat{t}) \approx \widehat{R}(\hat{x},\hat{t})$ is interpreted as the density of position states, the square integrability condition (10) for $\widehat{R}(\hat{x},\hat{t}) = \widehat{\psi^*}(\hat{x},\hat{t})\widehat{\psi}(\hat{x},\hat{t})$ is important, but, as we know the conventional quantum position and momentum states, $|x_0 >= \delta(x-x_0)$ and $|p >= (1/\sqrt{2\pi\hbar})\exp{(ip\,x/\hbar)}$, are not normalizable on real standard axes. The matter can be resolved in NSA by introducing one particle position and momentum states on the hyperfinite dimensional lattice,$\widetilde{|\hat{x} >}$ and $\widetilde{|\hat{p}\rangle}$, which satisfy (10). Although such states do not have analogous value on real line axes, they can be considered equivalent to conventional system of $|x\rangle$ and $|p\rangle$ states, which obey Dirac orthogonality conditions. We explain this further below.

The inner product can naturally be defined on the $\mathcal{ST}$-$\mathtt{lattice}$ as,

$$< \widehat{f},\widehat{g} >= L \sum_{x=-N^*l_0}^{N^*l_0} \widehat{f^*}(\hat{x})\widehat{g}(\hat{x}),$$

and the analogous value on the standard real axis is,

$$< f,g >= \mathbf{st}\big( < \widehat{f},\widehat{g} > \big).$$

The momentum and position operators on the $\mathcal{ST}$-$\mathtt{lattice}$ are naturally defined as,

$$\mathbf{P}\widehat{\psi}(\hat{x}) = -i\hbar\,(\widehat{\psi}(\hat{x}+L) - \widehat{\psi}(\hat{x})/L), \qquad \mathbf{X}\widehat{\psi}(\hat{x}) = \hat{x}\widehat{\psi}(\hat{x}).$$

The relationship $\mathbf{st}([\mathbf{P},\mathbf{X}]\ \widehat{\psi}(\hat{x})) = -i\,\hbar\,(\mathbf{st}\widehat{\psi}(\hat{x}))$ ensures the commutation relation between position and momentum operators on the standard real axis. As explained before, momentum is a wave property thus the momentum line can become discrete through infinitesimal spacing $\Delta p$ defined by the De-Broglie relation, $p = h/\lambda$, which comes from wave equation solutions on the real axis (for similar treatments e.g. see [1] and [2]),

$$\Delta p = h/\lambda_{max} = h/(2N^*l_0) = \pi\hbar/(N^*l_0).$$

Thus the momentum space is given by,

$$[\mathbb{Z}_{2(M^*)+1}]_{\text{Momentum}} = \{-N^*(\frac{\pi\hbar}{l_0}), \cdots, \frac{-\pi\hbar}{N^* l_0}, 0, \frac{\pi\hbar}{N^* l_0}, \cdots, N^*(\frac{\pi\hbar}{l_0})\},$$

The length of momentum line is equal to $(2N^*)\frac{\pi\hbar}{l_0} = \frac{h}{L}$. Since on real axes we are unable to define square integrable single particle position and momentum states, the normalization condition is changed using Delta Dirac distribution. This procedure can be represented on the $\mathcal{ST}$-lattice as,

*Dirac formalism for position and momentum states and orthogonality conditions:*

$$\delta_1(\hat{x}) = \begin{cases} \frac{1}{L} & \text{if } \hat{x} = 0, \\ 0 & \text{if } \hat{x} \neq 0. \end{cases}, \qquad \delta_2(\hat{k}) = \begin{cases} \frac{N^* l_0}{\pi\hbar} & \text{if } \hat{k} = 0, \\ 0 & \text{if } \hat{k} \neq 0. \end{cases},$$

$$|\hat{x}> \equiv \widehat{v}_{\hat{x}}(\hat{x}') = \begin{cases} \frac{1}{L} & \text{if } \hat{x'} = \hat{x}, \\ 0 & \text{if } \hat{x'} \neq \hat{x}. \end{cases}, \qquad |\hat{p}> \equiv \widehat{u}_{\hat{p}}(\hat{x}) = \frac{\exp(\frac{i\hat{p}\,\hat{x}}{\hbar})}{\sqrt{2\pi\hbar}},$$

$$< \widehat{v}_{\hat{x}}, \widehat{v}_{\hat{x'}} >= \delta_1(\hat{x} - \hat{x'}), \qquad < \widehat{u}_{\hat{p}}, \widehat{u}_{\hat{p}'} >\approx \delta_2(\hat{p} - \hat{p}'),$$

$$< \hat{x}|\hat{p} > =< \widehat{v}_{\hat{x}}(\hat{x}'), \widehat{u}_{\hat{p}}(\hat{x}') >= \widehat{u}_{\hat{p}}(\hat{x}) = \frac{\exp(\frac{i\hat{p}\,\hat{x}}{\hbar})}{\sqrt{2\pi\hbar}},$$

$$\widehat{\psi}(\hat{x}) = \sum_{\hat{x}' \in [\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} L\,\widehat{\psi}(\hat{x}')\,\widehat{v}_{\hat{x}}(\hat{x}'),$$

$$\widehat{\psi}(\hat{x}) \approx \sum_{\hat{p} \in [\mathbb{Z}_{2(M^*)+1}]_{\text{momentum}}} (\Delta p)\ \widehat{g}(\hat{p})\,\widehat{u}_{\hat{p}}(\hat{x}). \tag{A.1}$$

On the $\mathcal{ST}$-**lattice** we are able to define normalizable position and momentum states for a single particle, $\widetilde{|\hat{x}>} = \sqrt{L}\ |\hat{x}>$ and $\widetilde{|\hat{p}>} = \sqrt{\Delta p}\ |\hat{p}>$. Such states can only be defined on the $\mathcal{ST}$-**lattice** and the values related to such states vanish on standard real space and time lines.

*One particle normalized position and momentum states,*

$$\widetilde{|\hat{x}>} = \widetilde{v}_{\hat{x}}(\hat{x}') = \begin{cases} \frac{1}{\sqrt{L}} & \text{if } \hat{x}' = \hat{x}, \\ 0 & \text{if } \hat{x}' \neq \hat{x}. \end{cases} \qquad \widetilde{|\hat{p}>} = \widetilde{u}_{\hat{p}}(\hat{x}) = \frac{\exp(\frac{i\hat{p}\,\hat{x}}{\hbar})}{\sqrt{2N^* l_0}},$$

$$< \widetilde{v}_{\hat{x}}, \widetilde{v}_{\hat{x}'} >= \widetilde{\delta}_{\hat{x},\hat{x}'}, \qquad < \widetilde{u}_{\hat{p}}, \widehat{u}_{\hat{p}'} >\approx \widetilde{\delta}_{\hat{p},\hat{p}'},$$

$$\widetilde{|\hat{x}>} = \sqrt{L}\ |\hat{x}> = \sqrt{\frac{l_0}{N^*}}\,|\hat{x}>, \qquad \widetilde{|\hat{p}>} = \sqrt{\Delta p}\ |\hat{p}> = \sqrt{\frac{\pi\hbar}{N^* l_0}}\,|\hat{p}>,$$

$$\widetilde{< \hat{x}|\hat{p} >} = \sqrt{L\,\Delta p} < \hat{x}|\hat{p} >,$$

$$\widehat{\psi}(\hat{x}) \approx \sum_{\hat{p}\,\in[\mathbb{Z}_{2(M^*)+1}]_{\text{Momentum}}} \sqrt{\Delta p}\;\widehat{g}(\hat{p})\;\widetilde{u}_{\hat{p}}(\hat{x}),$$

$$\widehat{\psi}(\hat{x}) = \sum_{\hat{x}'\in[\mathbb{Z}_{2(M^*)+1}]_{\text{space}}} \sqrt{L}\;\widehat{\psi}(\hat{x}')\;\widetilde{v}_{\hat{x}}(\hat{x}'). \tag{A.2}$$

where $\widetilde{\delta}_{\hat{x},\hat{x}'}$ is Kronecker delta ($\widetilde{\delta}_{\hat{x},\hat{x}'}$ is equal to 1 if $x = x'$ and otherwise zero) on the space lattice and similarly $\widetilde{\delta}_{\hat{p},\hat{p}'}$ is Kronecker delta on the momentum lattice line.

*B. Quantum field theory and PBM interpretation*

The PBM model for particles and waves is consistent with the elements of the Fock space of $QFT$, as mentioned earlier. There has been debate over particle or field ontology or neither of them for $QFT$ (e.g. see [21] [28], [22], [19], [3], [12], [14] and [4]). Particle ontology of $QFT$ is, for example, based both on findings that charge and mass etc. are discrete and in addition observation of particle traces in bubble chambers. The fact that the Fock space is discrete and the particle number operator is well defined (e.g. for non-interacting fields) suggests one to one correspondence between particles and elements of the Fock space [28]. Nevertheless, different arguments have been presented against particle ontology, which can be divided into three parts, identity, unitary inequivalent representations and localizibility which we briefly re-examine here using the PBM model.

*1- Identity and individuality*

Although particles are countable, this characteristic alone is not enough to define a particle; it seems one also needs individuality. There are arguments over individuality of identical particles, as they are indistinguishable. This problem has its roots in $QM$ which is extended to $QFT$. In the PBM model particles are distinguished on the $\mathcal{ST}$-**lattice** since they all have distinct paths but based on the assumption of particle infinite speed in the $\mathcal{ST}$-**lattice**, the particles on the real space time axes become indistinguishable. The $\mathcal{BM}$ model, which assumes trajectories for particles, takes the particle traces in trace chambers as evidence of particle individuality. This is also true in the PBM model but through a different process of continual wave function collapses where particles go through every collapsed wave as discussed in 2.3, but this brings us to another problem which is localizibility.

*2- Unitary inequivalent representations*

The argument over occurrence of UIRs is against both particle and field ontology, by way of questioning the use of Fock space for interacting field theory. While, in non-interacting fields, the particle number operator is well defined and free fields can be equivalently described with Fock space, it is not true for interacting fields. As Haag has shown (see [19]) free fields and non-interacting field theory are unitarily inequivalent. The particle number operator can not be defined at any stage in interacting field theory. This creates problems for particle ontology, which assumes correspondence between elements of Fock space and particles, in addition to field ontology, since functional field theory is unitarily equivalent to Fock space. For this reason Bain, in [3], suggested that only asymptotically free states, i.e. states very long before or after a scattering interaction, have a Fock representation that allows for an interpretation in terms of countable quanta. This suggestion is rated by Fraser, in [14], as an unsuccessful last final attempt to save a quanta interpretation of $QFT$ because it is ad hoc and can not even show that at least something similar to the free field total number operator exists for *finite times*, i.e. between the asymptotically free states. Bain, in [4], points out that the reason why there is no total number operator in interacting relativistic quantum field theories is that this would require an *absolute space-time* structure. Taking the above into consideration, we suggest the following solution for this problem within the PBM framework. As is known when we look at smaller and smaller length scale (thus more energy and equivalently smaller time intervals) in $QFT$, due to the uncertainty principle, more and more intermediate particles (as well as various outcomes) appear in Feynman diagrams. In PBM model one can assume beyond any energy cut off of renormalization and thus beyond any finite time, there are infinitesimal time intervals (before and after interactions) where particles are only described by free fields. In other words, we admit that Fock space is a privileged representation since on the hyperfinite dimensional lattice, particles are described by free fields, except for the times where interactions are taking place.

*3- Localizibility*

There are several results concerning the impossibility of particle localization in $QFT$ one of which is the Malament result ([22], [20]) which formulates a no-go theorem to the effect that a relativistic quantum theory of a fixed number of particles predicts a zero probability for finding a particle in any spatial set, provided four conditions are satisfied, namely concerning translation covariance, energy, localizibility and locality. The Malament localizibility condition asserts that, a particle can not be found in two disjoint spatial sets at the same time. There have been several criticisms of the conclusion from the Malament theorem, one of which is that it assumes a fixed number of particles (only valid for free fields). Even though, until now, such results about non-localizibility in $QFT$ can not be considered complete in order to rule out particle ontology of $QFT$, they can reflect the difficulty in accounting for localizibilty in the $QFT$ framework. In the PBM model a particle can not be found in two disjoint spatial sets at the same time on the $\mathcal{ST}$-**lattice**. However, this is not true on real standard axes because of infinite speeds and there is a possibility of finding a particle in two disjoint spatial sets at the same time on standard real axes. In other words, the definition of particles in the PBM model does not satisfy the Malament Localizibilty condition. Taking into consideration the above results about non-localizibility of $QFT$ for free fields, a conclusion can be drawn for PBM that; the localization, which is the same as wave function reduction (collapse), is a phenomenon which happens on hyperreal time and space axes. As a result the wave function collapse in PBM is something beyond both $QM$ and $QFT$ which are defined and observed on real space time axes. In fact, the decoherence (which is the first step in the collapse process) can naturally be explained by $QFT$ (for example, heat exchange in the environment leads to the leaking of quantum information), but the final step, which is the random particle jump that finalizes the collapse, happens on hyperreal axes and we just perceive its evidence on standard real axes (e.g. particle traces in a bubble chamber).

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