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Original Paper

Hidden Variables Foundation of Matrix Mechanics

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Abstract: A preliminary hidden variables matrix mechanics treatment of the harmonic oscillator has been previously presented based on classical endogenous periodic motion. This work extends to incorporating the model into the mathematics of matrix mechanics. Although initially motivated by EPR-Bell analysis, the proposed model is based on re-examining the physical assumptions of Heisenberg and Born. All assumptions are maintained except for Bohr's state-to-state instantaneous transition which has been experimentally invalidated, and Heisenberg's non-path postulate which is replaced by classical endogenous periodic paths. Matrix elements of standard matrix mechanics are modified to replace transition amplitudes by transition paths. The redefined elements generate eigenvalues-eigenstates which then characterise eigenpaths. Since the endogenous motion averages out over a cycle it is unseen by the wave function. Nevertheless, mathematical equivalence with position and momentum non-commutation in Schrodinger operators is preserved. The modified matrix mechanics is shown to be mathematically equivalent to that of Born-Jordan reproducing all standard results. Generic quantum equations of motion are obtained following the quantization procedures of Born-Jordan and Dirac's Poisson Bracket equation. These new relations meet the benchmark criteria of reproducing conservation of energy and the quantum frequency condition. Since the endogenous paths are ontologically classical no radical metaphysical interpretations are needed for spatial-temporal movement. Quantum randomness is not explained by the proposed model but is attributed to endogenous structures of quantum matter.

Keywords: Quantum foundations, hidden variables, matrix mechanics, quantum Poisson Bracket, Bell inequalities, EPR-conundrum

1. Introduction

Bell inequalities which were motivated by the Einstein-Podosky-Rosen (EPR) paradox are central for exploring questions on quantum mechanics (QM) foundations [1]. A consensus on the *experimentally verified* inequalities violation is that no locally causal hidden variables (HV) theory is possible. Nevertheless, there are alternative views that violation can be based on contextuality (a significant alternative) or possibly superdeterminism, Bohr complementarity and/or dependent on descriptions used [2-8 (see ref.)]. As previously discussed, other interpretations have also been proposed, including Bell's own work with a free particle position-momentum configuration where there is no contradiction with QM [1,

9]. It is worth noting Bell's own conclusion on his later work: "*with the wave function (original EPR), then, there is no non-locality problem when the incompleteness of the wave function description is admitted".*

Interpretations of QM and EPR-Bell type analysis have dominated the foundations question, while exploring actual HV models as an alternative approach, has remained peripheral. Notwithstanding insights into the nature of QM emulating from Bell analysis, it is nevertheless restricted. Experiments confirm that the non-quantum inequalities are incorrect not why they are incorrect. No empirical evidence has emerged ruling out hidden variables, nor are such variables ruled out by inequalities violation. Furthermore, any conclusion is conditional on the assumption that all possible assumptions have been identified. As has already been stated, Bell's theorem "merely shows that a hidden variables theory *which fulfils all the assumptions of the theorem* is ruled out by observation" [4]. Whether QM is based on a deeper structure or maintaining the status quo, remains a matter of choice. If incompleteness is the chosen option, it seems reasonable to explore *actual* models.

Preliminary matrix mechanics (MM) hidden variables model based on a classical ontology was presented for the harmonic oscillator (HO) aiming to establish a physical rationale [9]. The primary aim of this work is to show the model can be incorporated into the *mathematics* of matrix mechanics.

EPR raised the possibility of HV, while Bell focused on whether such theories must be non-local. Bell was motivated by the apparent contradiction between von Neumann's completeness proof, that no HV could reproduce the results of QM, and Bohm - de Broglie's *actual* (non-local) theory which did.

By questioning the completeness of the wave function EPR concentrated attention on wave mechanics. However, the foundation question can also be approached independent of any opinion on Bell and EPR by re-examining the basic physical assumptions of quantum mechanics. If QM is indeed incomplete it is plausible to consider the possibility of limitations in the theory's basic assumptions. In formulating MM, Heisenberg and Born introduced postulates guided by the criteria of maintaining consistency with classical mechanics [10, 11]. Dirac was likewise guided by the same criteria [12]. Clarity of foundational assumptions makes MM simpler to re-examine.

As is now legend, Heisenberg initially attempted to mathematically describe atomic behaviour on the basis of classical paths. Concluding the approach was futile, he took the radical step of dispensing altogether with generic space and time trajectories opting instead for a description based solely on empirical observables. Born and also Dirac noted the essential feature of Heisenberg's mathematics was the non-commutation of the product rule for quantum observables. The question arises whether Heisenberg's non-path postulate is *essential*. At least for matrix mechanics it is found that it is unnecessary to take such a radical step.

The relevant Heisenberg and Born postulates are briefly repeated, ensuring a more self-contained presentation. The generic non-commutation relation of a modified MM was previously obtained following similar reasoning to Born-Jordan (BJ) [9]. In this presentation all basic relations are obtained from Dirac's quantum Poisson Bracket (PB) equation together with Born-Jordan. Critically, the new relations meet the benchmark criteria of reproducing conservation of energy and the frequency condition. Mathematical equivalence with standard MM is maintained.

For an actual HV theory reproducing QM results is seen to be the stronger criteria of judgement. An inequalities-type conclusion that no HV theory subject to particular constraints can reproduce the results of QM is invalided by an actual such theory which does.

Nevertheless, EPR-Bell remains a *guiding* analysis. Mermin's EPR-conundrum will be briefly considered in relation to the proposed model.

2. Physical Postulates

Heisenberg's most critical assumption was to reject continuous path movement on the basis of operational non-measurability, and also that an orbital theory did not reproduce experiment - it is not "*possible to associate the electron with a point in space, considered as a function of time*". Rejection of orbital-type movement does not however necessitate rejection of all other forms of endogenous motion.

What was termed a kinematic re-interpretation, Heisenberg retained empirically observable transition amplitudes which Born mathematically represented as matrix elements. Working from the experimentally verified frequency condition Heisenberg concluded that amplitudes were a function of two indices representing the initial and final states of a transition. Bohr had previously introduced the assumption that state-to-state transitions were instantaneous. This assumption has been found to be invalid. Experiments on atomic systems show duration intervals in the order of attoseconds [13-14].

Heisenberg's non-path postulate will be replaced by the assumption that the quantum particle entity can at all times *be associated* with a point in physical space which evolves continuously in space and time. Placing the quantum object in a space and time platform during the attoseconds transition remains the simplest explanation, circumventing the need for complex metaphysics. A micro-path, likewise dependent on two indices, can then be associated with a transition. Physically, matrix elements can be interpreted to refer to classical paths of finite duration. With this re-definition the position matrix would refer to an ensemble of time-dependent transition positions describing particle-paths. Based on experimental duration intervals, the proposed paths will be on the scale of the electron wavelength for atomic systems.

Physically, it has to be assumed *that the ensemble is a feature of an individual object.* Associating an ensemble with a single entity may seem an implausible departure from the classical definition of a particle. However, if all ensemble information is not contained by a single entity it becomes difficult to see how protective measurements, which have been performed experimentally, would be possible [15-16]. Notwithstanding some controversy, the single particle ensemble assumption thereby has empirical justification. Under this assumption however quantum matter cannot be seen as point particles without internal definition. W*hat is a particle?* remains an interesting question [17].

That QM can be based on an endogenous substructure has been previously proposed [16]. With the preliminary analysis a quaternion representation in 2×2 matrix was introduced to describe a unit transition. That assumption will continue but subject to development. Heisenberg introduced a number of other fundamental postulates which will continue to apply, as previously discussed [9]. Obviously, the proposed model adheres to the statistical interpretation of QM [18].

Recently, von Neumann's theorem has been re-interpreted as showing that HV theories cannot represent their quantities as Hermitian operators, rather than the original completeness conclusion. Whether the proof was actually about completeness is a matter of historical conjecture. A re-assessment of Kochen-Specker has found that contextuality is not imposed on possible HV theories [19]. While the proposed model is consistent with both theorems its validity is based on the stronger condition of reproducing QM [9].

The legendary discoveries of Plank and Einstein established that the generic quantum process is a discontinuous, discrete energy transaction between an energy-quanta and a particle in oscillations. This process is the essence of Heisenberg and Born quantization

procedure with the modification that oscillations are omitted by the non-path postulate, replaced by a more general concept of particle energy-state. By re-introducing oscillatory motion the proposed model aligns with the original concepts.

3. Matrix Mechanics

Born accepted the basics of Heisenberg's mathematics: applicability of classical equations of motion, non-commutation of the product rule for quantum variables, as well as the frequency and quantum conditions:

$$\omega(n,m) + \omega(m,k) = \omega(n,k) \tag{1a}$$

$$BJ: \sum_k [p(n,k)q(k,n) - q(n,k)p(k,n)] = (-i\hbar) \tag{1b}$$

$$Heisenberg: \quad \sum_k \omega(k,n)|q(n,k)|^2 = \frac{\hbar}{2m} \tag{1c}$$

Non-commutation inspired Born to mathematically represent quantum observables by matrices where elements represent transition amplitudes. Physically however a matrix representation is seen here as a consequence of the ensemble assumption. Replacing Heisenberg's transition amplitudes with endogenous motion, assuming such motion is the same as that of the harmonic oscillator, gives the ensemble quantities in BJ (notation) infinite order matrix as [9]:

$$\boldsymbol{q} = \boldsymbol{q_1} + \boldsymbol{q_2} = \left(q_1(n,m)e^{2\pi i\upsilon(n,m)t} + iq_2(n,m)e^{-2\pi i\upsilon(n,m)t}\right) \tag{2a}$$

$$\boldsymbol{p} = \boldsymbol{p_1} + \boldsymbol{p_2} = (p_1(n,m)e^{2\pi i\upsilon(n,m)t} + ip_2(n,m)e^{-2\pi i\upsilon(n,m)t}) \tag{2b}$$

If the $\boldsymbol{i}$ factor in the second terms is omitted the final results will be the same provided definitions remain consistent. Matrix elements which describe the individual transition path are:

$$q_{end}(n,m,t) = q_1(n,m)e^{2\pi i\upsilon(n,m)t} + iq_2(n,m)e^{-2\pi i\upsilon(n,m)t} \tag{3}$$

Relations (2) give two definitions of position (and momentum): the classical endogenous position represented by constituent matrix elements which is not defined in Born-Jordan and the quantum ensemble position represented by the infinite order matrix. Each is governed by different algebra implying different mathematical (including measurability as a geometric-mathematical property) and so different physical properties [9].

It is important to clarify that the following equations define relations between *ensemble* quantities subject to Heisenberg's three conditions. Element components of (3) corresponding to BJ will initially be referred to as transition amplitudes in keeping with Heisenberg's definition. However, since amplitudes and paths appear to be different there is a potential inconsistency in the meaning of the ensemble quantities. This issue will be addressed. That endogenous position may be a complex number raises the question of its physical meaning. Complex number particle trajectories and their meaningful physical association with QM have been proposed [20]. For Heisenberg's measurable-only quantities endogenous paths are nevertheless found to be represented by real numbers.

Born-Jordan introduced a diagonal matrix **W,** whose elements are energies consistent with the frequency condition to provide connection with experiment. Also introduced is a general matrix function **g** = **g(p, q)** which in the modified form is:

$$\boldsymbol{g} = \boldsymbol{g_1} + \boldsymbol{g_2} = (g_1(n,m)e^{2\pi i\upsilon(n,m)t} + ig_2(n,m)e^{-2\pi i\upsilon(n,m)t}) \tag{4}$$

Mathematically, it may seem that this relation should also include product cross terms emerging from the two components of (2a, b). Physically however, Born-Jordan asserted that all quantities – position, momentum and functions of both – must have the same time factors. In which case, the general **g(p, q)** must have the same form as position and momentum. This critical assertion will continue to be addressed. Differentiating gives:

$$\dot{\boldsymbol{g}} = 2\pi i(\upsilon(n,m)\left(g_1(n,m)e^{2\pi i\upsilon(n,m)t} - ig_2(n,m)e^{-2\pi i\upsilon(n,m)t}\right)) \tag{5}$$

Following BJ gives:

$$\dot{\boldsymbol{g}} = \left(\frac{i}{\hbar}\right)\{[\boldsymbol{W}, \boldsymbol{g_1}] - [\boldsymbol{W}, \boldsymbol{g_2}]\} \tag{6}$$

For the Hamiltonian energy-matrix the relation is then:

$$\dot{\boldsymbol{H}} = (\frac{i}{\hbar})\{[\boldsymbol{W}, \boldsymbol{H_1}] - [\boldsymbol{W}, \boldsymbol{H_2}]\} \tag{7}$$

If $\dot{\boldsymbol{g}} = 0$ BJ show that the standard general matrix must be diagonal. From relation (5) the conditions for non-diagonal elements are:

$$for\ n \neq m, \upsilon(n,m) \neq 0: \tag{8a}$$

$$g_1(n,m)e^{2\pi i\upsilon(n,m)t} - ig_2(n,m)e^{-2\pi i\upsilon(n,m)t} = 0 \tag{8b}$$

$$so\ that: \quad g_1(n,m) = g_2(n,m) = 0 \tag{8c}$$

Condition (8c) establishes that if $\dot{\boldsymbol{g}} = 0$ then **g** = **g(p, q)** is diagonal which defines the condition for conserved quantities.

From the periodicity condition Born-Jordan derived Heisenberg's quantum condition (1b), which applies only for diagonal elements. Following similar reasoning gives the modified relation:

$$\sum_k \{[p_1(n,k)q_1(k,n) - q_1(n,k)p_1(k,n)] - [p_2(n,k)q_2(k,n) - q_2(n,k)p_2(k,n)]\} = (-i\hbar) \tag{9}$$

Importantly, component cross terms cancel. This property ensures the relation is consistent with the frequency condition (1a), as required. Born-Jordan then established all off-diagonal elements are zero. Following similar steps gives the same results for the modified matrices, where the Hamiltonian takes the general form (4). Accordingly, the modified quantum relation becomes [9]:

$$[\boldsymbol{q_1}, \boldsymbol{p_1}] - [\boldsymbol{q}_2, \boldsymbol{p}_2] = i\hbar \tag{10}$$

Relations (9) and (10) remain the same if the $\boldsymbol{i}$ factor in (2) is omitted. This equation refers to ensemble position and momentum transition amplitudes matrices not directly to transition paths. The same relation is also obtained following Dirac [12].

Like Born, Dirac accepted the basics of Heisenberg's mathematics to define a general quantum algebra based on non-commutation and the frequency condition. Dirac also accepted the basic classical Heisenberg assumption that variables are still represented by Fourier series. Accordingly, variables x (and y) are defined as (Dirac notation):

$$classical: \ x(t) = \sum x_\alpha e^{i(\alpha\omega)t} \tag{11}$$

Focusing on components, the general component product term of the product of the two variables is:

$$classical: \quad ae^{i(\alpha\omega)t}be^{i(\beta\omega)t} = abe^{i(\alpha+\beta)\omega t} \tag{12a}$$

$$quantum: \ a(n,k)e^{i\omega(n,k)t}b(k,m)e^{i\omega(k,m)t} = ab(n,m)e^{i\omega(n,m)t} \tag{12b}$$

Both non-commutative multiplication and critically the frequency condition (1a) are incorporated into the quantum relation (12b).

Dirac began by defining the non-commutation structure of the QM variables based on their two indices. By applying the classical asymptotic condition a difference expression was constructed for each variable leading to differential terms. The quantum Poisson Bracket equation followed as [12, 21(more detailed)]:

$$xy - yx = i\hbar\sum\left\{\frac{\partial x}{\partial q}\frac{\partial y}{\partial p} - \frac{\partial y}{\partial q}\frac{\partial x}{\partial p}\right\} = i\hbar\{PB\}_{classical} \tag{13}$$

Since the following relations are obtained following Dirac only an outline identifying the appropriate changes will be presented. Accordingly, the modified Dirac x-variable in analogy with (11) for a particular (n, m) component will take the form:

$$quantum: \ x(n,m,t) = x_1(n,m)e^{i\omega(n,m)t} + x_2(n,m)e^{-i\omega(n,m)t} \tag{14a}$$

$$classical: \ x^\alpha(t) = x_1^\alpha e^{i(\alpha\omega)t} + x_2^\alpha e^{-i(\alpha\omega)t} \tag{14b}$$

Corresponding form can be expressed for the y-variable. The two terms of the quantum component (14a) describe the transition paths introduced by assumption as matrix elements of the ensemble position matrix. Initially, no assumptions are made about the relation between the two terms, nor is Heisenberg's classical assumption (11) of a single series assumed. Relation (14b) is chosen as the corresponding classical expression using the usual Fourier series basis functions. Clearly (14) which generates two series is mathematically different from the single series (11). While the second series is a complex conjugate, the two added terms of (14) still form the same general (n, m) component as (12). Since the matrix elements represent a different quantity to those of BJ the mathematical difference is expected. It will be shown however, that all results can be obtained by commencing from Heisenberg's initial single Fourier assumption.

Dirac's quantization procedure involves two steps: constructing the non-commutation relations and using the classical asymptotic condition to obtain the PB expressions. Despite different quantities, Dirac's mathematics is also applicable to the modified relations (14). As with the general matrix (4) it may seem that relation (14a) should include cross terms. This important issue will be addressed. For position and momentum however, there are no cross terms in which case relation (14) is appropriate. The general quantum component for either of the two variables is then the sum of two terms:

$$a_1(n,m)e^{i\omega(n,m)t} + a_2(n,m)e^{-i\omega(n,m)t} \tag{15}$$

Accordingly, the corresponding general component for the quantum product term for the product of the two variables is then:

$$a_1b_1e^{i\omega(n,k)t} + a_2b_2e^{-i\omega(n,k)t} + a_1b_2e^{i\omega_1(n,m,k)t} + a_2b_1e^{i\omega_2(n,m,k)t} \tag{16}$$

Critically, the cross term frequencies i.e. $\omega_1(n,m,k), \omega_2(n,m,k)$, violate the quantum frequency condition (1a), unlike the Dirac product (12b), and are therefore non-QM. For the general matrix (4) potential cross terms likewise violate the frequency condition and are also non-QM. For (13) the LHS is:

$$[x,y] = [x_1 + x_2, y_1 + y_2] = [x_1,y_1] + [x_2,y_2] + [x_1,y_2] + [x_2,y_1] \tag{17}$$

Again, the two cross terms will violate the frequency condition. Generic relations must be obtained such that non-QM cross terms are excluded.

Dirac introduced the following commutation expression together with a product term for x and y variables for the general (n, m) component:

$$x(n,n-\alpha)y(n-\alpha,n-\alpha-\beta) - y(n,n-\beta)x(n-\beta,n-\alpha-\beta) \tag{18a}$$

$$product\ term\text{: } x(n-\beta,n-\beta-\alpha)y(n-\alpha,n-\alpha-\beta) \tag{18b}$$

$$where\ x(n,n-\alpha) \Rightarrow x_{\alpha_\kappa}\ ;\ y(n,n-\beta) \Rightarrow y_{\beta_\kappa} \tag{18c}$$

Relations (18c) define the asymptotic classical analogy for the quantum variables. Similar asymptotic relations apply to the other terms of (18a). The product term is added and subtract from expression (18a). Applying the asymptotic conditions, the resulting expression leads to Dirac's differential (via the difference) relation as a function of the action (κ is identified with the action variable J):

$$h\sum\{\beta\frac{\partial x_{\alpha_\kappa}}{\partial\kappa}y_{\beta_\kappa} - \alpha\frac{\partial y_{\beta K}}{\partial\kappa}x_{\alpha_\kappa}\} \tag{19a}$$

$$where\ for\ x\text{:}\ \ \Delta x_{\alpha_\kappa} = \{x_{\alpha_\kappa}(\kappa) - x_{\alpha_\kappa}(\kappa-\Delta\kappa)\} \tag{19b}$$

$$with\ \Delta\kappa = h\Delta n \Rightarrow h\beta \tag{19c}$$

A corresponding intermediary expression to (19b) is defined for the y-variable.

Relation (18a) is modified with each quantum variable becoming the sum of two terms with indices unchanged while the product term is also modified giving:

$$x \Rightarrow x_1(n,n-\alpha) + x_2(n,n-\alpha) \tag{20a}$$

$$x(n-\beta,n-\beta-\alpha)y(n-\alpha,n-\alpha-\beta) \Rightarrow$$

$$\{x_1(n-\beta_1, n-\beta_1-\alpha) - x_2(n-\beta_2, n-\beta_2-\alpha)\}\times$$
$$\{y_1(n-\alpha_1, n-\alpha_1-\beta) - y_2(n-\alpha_2, n-\alpha_2-\beta)\} \quad (20b)$$
$$by\ def^n\text{: } \beta = \beta_1+\beta_2\ and\ \alpha = \alpha_1+\alpha_2 \quad (20c)$$

It might seem the β-values for the x-term (and α-values for the y-term) should be equal. While it is found that $\beta_1 = \beta_2$ this condition will not be assumed. The product term (20b) is again added and subtracted to the modified expression of (18a). Assuming the asymptotic classical analogy (18c) applies to both components (both subject to same criteria) the expression corresponding to the first term of (19a) is:

$$\{[x_1(n, n-\alpha) - x_1(n-\beta_1, n-\beta_1-\alpha)] +$$
$$[x_2(n, n-\alpha) - x_2(n-\beta_2, n-\beta_2-\alpha)]\}y_{\beta\kappa} \quad (21a)$$
$$\Rightarrow \{(\Delta x_1^{\alpha_\kappa})_{\Delta n=\beta_1} + (\Delta x_2^{\alpha_\kappa})_{\Delta n=\beta_2}\}y_{\beta\kappa} \quad (21b)$$
$$\Delta x_1^{\alpha_\kappa} = \{x_1^{\alpha_\kappa}(\kappa) - x_1^{\alpha_\kappa}(\kappa-\Delta\kappa_1)\}\ and\ \Delta x_2^{\alpha_\kappa} = \{x_2^{\alpha_\kappa}(\kappa) - x_2^{\alpha_\kappa}(\kappa-\Delta\kappa_2)\} \quad (21c)$$
$$with\ \Delta\kappa_1 = h\Delta n\ \Rightarrow h\beta_1\ and\ \Delta\kappa_2 = h\Delta n\ \Rightarrow h\beta_2 \quad (21d)$$

That the action incremental values in (21c) are not equal follows from assuming the β-values are unequal. This condition relates directly to the quantum energy changes. Following Dirac gives:

$$(h\Delta n)\left(\frac{\Delta x_{\alpha_\kappa}}{\mathrm{h\Delta n}}\right)_{\Delta n=\beta} \Rightarrow (h\beta)\{\frac{\beta_1}{\beta}\left(\frac{\Delta x_1^{\alpha_\kappa}}{\mathrm{h\Delta n}}\right)_{\Delta n=\beta_1} + \frac{\beta_2}{\beta}\left(\frac{\Delta x_2^{\alpha_\kappa}}{\mathrm{h\Delta n}}\right)_{\Delta n=\beta_2}\} \quad (22a)$$
$$\Rightarrow h\beta\{\varepsilon_1\frac{\partial x_1^{\alpha_\kappa}}{\partial\kappa} + \varepsilon_2\frac{\partial x_2^{\alpha_\kappa}}{\partial\kappa}\} \quad (22b)$$
$$\varepsilon_1 = \frac{\beta_1}{\beta}\ and\ \varepsilon_2 = \frac{\beta_2}{\beta}\ ;\ \varepsilon_1+\varepsilon_2 = 1 \quad (22c)$$

These relations lead to:

$$(h\beta)\left(\frac{\partial x_{\alpha_\kappa}}{\partial\kappa}\right)_{Dirac} \Rightarrow h\beta[\varepsilon_1\frac{\partial x_1^{\alpha_\kappa}}{\partial\kappa} + \varepsilon_2\frac{\partial x_2^{\alpha_\kappa}}{\partial\kappa}] \quad (23)$$

Using the classical definition for the y-variable corresponding to (11a) Dirac introduced the relation:

$$2\pi i\beta y_\beta\, e^{i\beta\omega t} = \frac{\partial}{\partial\omega}\{y_\beta\, e^{i\beta\omega t}\}\ \ where\ \omega = \omega t/2\pi \quad (24)$$

Substituting (24) into (19a) completes the first Dirac term. Repeating with interchanging x and y variables completes the second Dirac term of (19a). The resulting expression represents the (n, m) component of $(xy-yx)$.

The corresponding modified classical definition for the y-variable is:

$$2\pi i\beta[y_1^{\beta_\kappa}e^{i\beta\omega t} + y_2^{\beta_\kappa}e^{-i\beta\omega t}] = \frac{\partial}{\partial\omega}\{y_1^{\beta_\kappa}e^{i\beta\omega t} - y_2^{\beta_\kappa}e^{-i\beta\omega t}\} \quad (25a)$$
$$so\ that\text{: } (\frac{\partial}{\partial\omega}\{y_\beta\, e^{i\beta\omega t}\})_{Dirac} \Rightarrow \frac{\partial}{\partial\omega}\{y_1^{\beta_\kappa}e^{i\beta\omega t} - y_2^{\beta_\kappa}e^{-i\beta\omega t}\} \quad (25b)$$

Combining relations (20) to (25) and following Dirac leads to the modified corresponding first term of Dirac's relation (19a). By interchanging x and y variables and repeating steps leads to the second modified term. The resulting relation again represents the (n, m) component of $(xy-yx)$. Following Dirac, $(xy-yx)$ itself for p's and q's as canonical variables equals:

$$[x,y] = i\hbar\sum\{\frac{\partial(x_1-x_2)}{\partial q}\frac{\partial(\varepsilon_1y_1+\varepsilon_2y_2\}}{\partial p} - \frac{\partial(y_1-y_2)}{\partial q}\frac{\partial(\varepsilon_1x_1+\varepsilon_2x_2\}}{\partial p}\} \quad (26)$$

It is assumed that (22c) also apples to α-values. This modified equation defines a connection between quantum quantities and basis HV.

Position and momentum definitions (2a) and (2b) define the corresponding Dirac variables as:

$$x = \boldsymbol{q_1} - \boldsymbol{iq_2} \quad (27a)$$
$$y = \boldsymbol{p_1} - \boldsymbol{ip_2} \quad (27b)$$

With these definitions the imaginary constant has been introduced to mathematically distinguish the non-QM terms. Final results will be the same if the constant is omitted. The LHS of (26) becomes:

$$[x, y] \Rightarrow [\boldsymbol{q_1}, \boldsymbol{p_1}] - [\boldsymbol{q_2}, \boldsymbol{p_2}] - i[\boldsymbol{q_1}, \boldsymbol{p_2}] - i[\boldsymbol{q_2}, \boldsymbol{p_1}] \tag{28}$$

Substituting into the RHS gives:

$$i\hbar(\varepsilon_1\{\boldsymbol{q_1}, \boldsymbol{p_1}\}_{PB} + \varepsilon_2\{\boldsymbol{q_2}, \boldsymbol{p_2}\}_{PB} - i\varepsilon_2\{\boldsymbol{q_1}, \boldsymbol{p_2}\}_{PB} + i\varepsilon_1\{\boldsymbol{q_2}, \boldsymbol{p_1}\}_{PB}) \tag{29a}$$

$$which\ equals{:}\ =\ i\hbar f((\varepsilon_1, \varepsilon_2) \tag{29b}$$

$$where:\ f(\varepsilon_1, \varepsilon_2) = (\varepsilon_1 + \varepsilon_2 + i\varepsilon_1 - i\varepsilon_2) \tag{29c}$$

Each individual non-commutation term on the LHS (28) also equals its corresponding RHS (29a) individual term. For expression (29a) all PB's are equal to one giving (29c). Born-Jordan's quantization procedure differentiates the action integral w.r.t. the action, which means the derivative of the integral must then equal one, since $\frac{\partial J}{\partial J} = 1$[10]. To make the transition from classical to quantum frequency the derivative of the integral becomes a difference leading by correspondence to the quantum non-commutation bracket to the factor $(1/i\hbar)$. Since the modified relation (10) follows Born-Jordan its non-commutation constant is also one. Dirac commences by defining the non-commutation bracket leading to the PB which for canonical variables (p, q) equals one. Following Dirac's procedure the modified relation (29a) leads to a constant i.e. $f(\varepsilon_1, \varepsilon_2)$. However, Born-Jordan and Dirac quantisation procedures must be equivalent. In which case the Dirac modified relation constant must also equal one. Hence:

$$(\varepsilon_1 + \varepsilon_2) + i(\varepsilon_1 - \varepsilon_2) = 1 \tag{30}$$

Since the imaginary term must be zero the two component cross terms of (30) cancel meeting the requirement of the frequency condition. Leaving:

$$\varepsilon_1 = \varepsilon_2 = ½ \tag{31a}$$

$$giving\quad [\boldsymbol{q_1}, \boldsymbol{p_1}] - [\boldsymbol{q_2}, \boldsymbol{p_2}] = i\hbar \tag{31b}$$

Alternatively, relation (31a) follows directly by assuming equal β-values (and α-values) for the two components as a basic condition. Relation (31b) is obtained without Born-Jordan's physical assertion on the form of the general function (4).

To obtain the corresponding equations of motion **H(p, q)** and **g(p, q)** must likewise be based on Heisenberg's mathematics, which following Born-Jordan, is done by obtaining the general functions from the non-commutation relation (31b). Relation (31b) reduced to its constituent components can be split into:

$$[\boldsymbol{q_1}, \boldsymbol{p_1}] = ½i\hbar\ and\ [\boldsymbol{q_2}, \boldsymbol{p_2}] = -½i\hbar \tag{32}$$

Using relations (32) and following Dirac reproduces the modified Heisenberg version (1c) of the quantum condition.

Following Born-Jordan for each individual term individually leads to:

$$[\boldsymbol{q_1}, \boldsymbol{H_1}] + [\boldsymbol{H_2}, \boldsymbol{q_2}] = ½(i\hbar)\left[\dot{\boldsymbol{q_1}} + \dot{\boldsymbol{q_2}}\right] = ½(i\hbar)\dot{\boldsymbol{q}} \tag{33a}$$

$$where\ \boldsymbol{H}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{H_1}(\boldsymbol{p_1}, \boldsymbol{q_1}) + \boldsymbol{H_2}(\boldsymbol{p_2}, \boldsymbol{q_2}) \tag{33b}$$

The same relation can also be obtained following Dirac. By splitting (33a) into two individual component relations, and following Dirac for each gives:

$$\Delta H_1 = [H_1(n, n) - H_1(m, m) = ½\hbar\omega(n, m) \tag{34a}$$

$$\Delta H_2 = [H_2(n, n) - H_2(m, m) = ½\hbar\omega(n, m) \tag{34b}$$

$$and\ \Delta H = \Delta H_1 + \Delta H_2 = \hbar\omega(n, m) \tag{34c}$$

Relations (31a) and (31b) imply that the total incremental change in action variable $\Delta\kappa$ of Dirac relation (19b) becomes the sum of equal values for each component in relations (32), which results in the quantum half energy contributions of (34a, b), and the total energy change (34c).These relations meet the benchmark criteria of reproducing the frequency condition connecting frequencies with quantum energy differences.

For the general function $\boldsymbol{g}(\boldsymbol{p},\boldsymbol{q})$ following either Born-Jordan or Dirac gives:

$$[\boldsymbol{H_1},\boldsymbol{g_1}] = ½(-i\hbar)\dot{\boldsymbol{g_1}} \;\; and \;\; [\boldsymbol{H_2},\boldsymbol{g_2}] = ½(i\hbar)\dot{\boldsymbol{g_2}} \tag{35a}$$

$$[\boldsymbol{g_1},\boldsymbol{H_1},] + [\boldsymbol{H_2},\boldsymbol{g_2}] = ½(i\hbar)\left[\dot{\boldsymbol{g_1}} + \dot{\boldsymbol{g_2}}\right] = ½(i\hbar)\dot{\boldsymbol{g}} \tag{35b}$$

$$where\ \boldsymbol{g}(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{g_1}(\boldsymbol{p_1},\boldsymbol{q_1}) + \boldsymbol{g_2}(\boldsymbol{p_2},\boldsymbol{q_2}) \tag{35c}$$

Like the energy-matrix, the general function (35c) does not contain cross terms, which is a consequence of the frequency condition and Born's assumed physical property of the system. Mathematically, the classical Hamilton and the energy-matrix can then take different forms. Substituting $\boldsymbol{H}(\boldsymbol{p},\boldsymbol{q}) = g(\boldsymbol{p},\boldsymbol{q})$ relation (35a) gives conservation of energy. Following BJ $H_1 and\ H_2$ must be diagonal in which case the energy- matrix must also be diagonal. In summary:

$$\Delta H = \hbar\omega,\ \dot{H} = 0\ and\ \dot{H}\ is\ diagonal \tag{36}$$

These conditions are the three fundamental requirements of matrix mechanics.

Equivalence with Born-Jordan is established by comparing transition probabilities as measurable quantities. For the modified position matrix the transition probabilities averaged over a cycle are:

$$< |a(n,m)|^2_{modified} > = \ |q_1\,(n,m)|^2 + |q_2(n,m)|^2 \tag{37}$$

The equivalence with Born-Jordan follows as:

$$\left|q_{BJ}(n,m)\right|^2 \equiv \ < |a(n,m)|^2_{modified} > \equiv |q_1\,(n,m)|^2 + |q_2(n,m)|^2 \tag{38}$$

Relation (38) also identifies equivalence with the wave function:

$$< |a(n,m)|^2_{modified} > \equiv \ \left|q_{BJ}(n,m)\right|^2 \equiv \int \varphi_n x \varphi_n dx \tag{39}$$

Since the endogenous motion is averaged over one cycle to the transition amplitudes, it cannot be described by the wave function. Comparing non-commutation relations (32) with Born-Jordan and assuming equivalence gives:

$$[\boldsymbol{q_1},\boldsymbol{p_1}] = ½\left[\boldsymbol{q_{BJ}},\boldsymbol{p_{BJ}}\right] and\ [\boldsymbol{q_2},\boldsymbol{p_2}] = ½\left[\overline{\boldsymbol{q_{BJ}}},\overline{\boldsymbol{p_{BJ}}}\right] \tag{40a}$$

$$giving: \quad q_1\,(n,m) = \frac{q_{BJ}(n,m)}{\sqrt{2}}\ ;\ p_1\,(n,m) = \frac{p_{BJ}(n,m)}{\sqrt{2}} \tag{40b}$$

$$iq_2\,(n,m) = \frac{\overline{q_{BJ}(n,m)}}{\sqrt{2}};\ ip_2\,(n,m) = \frac{\overline{p_{BJ}(n,m)}}{\sqrt{2}} \tag{40c}$$

If the imaginary constant is omitted in (2) it will need to be omitted in (40c). It is important to clarify that equations (31b), (32) and equations (40) which follow, refer to transition amplitudes not paths, giving modified transition amplitudes (40b, c). These relations define a direct connection with Heisenberg's measurable-only quantities. Using relations (40b, c) with (9), the standard BJ quantum condition (1b) is obtained. Heisenberg's original version (1c) is also obtained following Dirac.

Position and momentum matrix elements which describe the endogenous trajectories become:

$$q(n,m,t) = \frac{q_{BJ}(n,m)}{\sqrt{2}}e^{i\omega(n,m)t} + \frac{\overline{q_{BJ}(n,m)}}{\sqrt{2}}e^{-i\omega(n,m)t} \tag{41a}$$

$$= \sqrt{2}[q_R cos\omega(n,m)t - q_I sin\omega(n,m)t] \tag{41b}$$

$$where\ q_R = Req_{BJ}(n,m)\ and\ \ q_I = Imq_{BJ}(n,m)$$

$$p(n,m,t) = \frac{p_{BJ}(n,m)}{\sqrt{2}}e^{i\omega(n,m)t} + \frac{\overline{p_{BJ}(n,m)}}{\sqrt{2}}e^{-i\omega(n,m)t} \tag{41c}$$

$$= \sqrt{2}[p_R cos\omega(n,m)t - p_I sin\omega(n,m)t] \tag{41d}$$

$$where\ p_R = Rep_{BJ}(n,m)\ and\ p_I = Imp_{BJ}(n,m)$$

$$BJ: \ \ q_{BJ}(n,m,t) = [q_R cos\omega(n,m)t - q_I sin\omega(n,m)t] + i[q_I cos\omega(n,m)t + q_R sin\omega(n,m)t] \tag{41e}$$

Using (41a, c) position and momentum matrices (2a) and (2b) can be obtained directly from standard MM. In which case, Heisenberg's measurable-only quantities also determine the endogenous motion. From relations (41b) and (41d) it is clear paths are represented by real numbers. Relation (41e) expresses the BJ transition amplitude in the same form as the paths although represented by complex numbers. A corresponding relation can be obtained for momentum.

Using BJ equivalence relation (40a) together with (31b) and the Schrodinger operators for position and momentum gives:

$$[\boldsymbol{q_1}, \boldsymbol{p}_1] - [\boldsymbol{q_2}, \boldsymbol{p_2}] \equiv \tfrac{1}{2}[x, -i\hbar D_x] - \tfrac{1}{2}[x, i\hbar D_x] \quad (42a)$$

$$giving\colon\ [x, -i\hbar D_x] = i\hbar \quad (42b)$$

Since the wave function does not describe the endogenous motion, the modified MM and Schrodinger equation are not physically equivalent. In which case, a more nuanced definition of wave function completeness may be required. Nevertheless, relation (42b) is mathematically equivalent to the corresponding Schrodinger operator relation for Heisenberg's measurable-only quantities, ensuring computation compatibility.

Omitting the imaginary constant in (27) and continuing with relations (28) and (29) gives:

$$[\boldsymbol{q}, \boldsymbol{p}] = [\boldsymbol{q_1}, \boldsymbol{p}_1] + [\boldsymbol{q_2}, \boldsymbol{p_2}] = 0 \quad (43a)$$

$$where\colon [\boldsymbol{q_1}, \boldsymbol{p}_1] - [\boldsymbol{q_2}, \boldsymbol{p_2}] \equiv \left[\boldsymbol{q_{BJ}}, \boldsymbol{p_{BJ}}\right] = i\hbar \quad (43b)$$

Both non-commutation relations (31b) and (32) are still obtained. Relation (43b) follows as a consequence of the mathematical equivalence between the modified and original Heisenberg-BJ quantum condition. Equivalence with Schrodinger operators (42) can also be obtained from (43b). Cross-terms in (43a) are not included as they cancel. Relation (43a), where matrix elements refer to transition paths, is not consistent with the quantum condition (1b), and so the periodicity condition, and is thereby not the quantum basis relation, unlike (10). There is not a corresponding relation for Schrodinger operators. Nevertheless, it appears to violate the uncertainty principle in that eigenvalue position and momentum are simultaneously measurable. The violation is however, a question of interpretation not experiment. Experimentally verified preparation and measurement formulations of the uncertainty principle, together with error-disturbance relations, refer to ensemble distributions and relations between such distributions [9 (see refs.)]. Distribution properties of QM are not in question. As Ballentine pointed out in the statistical interpretation of QM, uncertainty relations apply to distributions.

Complementarity is preserved by equation (42b) and so preserved as a property of the wave function, and by (43b) as a property of position and momentum transition amplitudes (consistent with BJ) but not with (43a), although all are underpinned by the same HV.

That matrix elements have different meaning creates a difficulty in interpreting the ensemble quantities, as identified. However, by relation (41c) BJ amplitudes can also be interpreted as describing complex transition paths. Born –Jordan and HV ensemble position and momentum will then have the same physical meaning. Being complex numbers BJ quantities cannot directly be associated with physical quantities. That is not the case however with HV quantities which are represented by real numbers. Computability remains the same as seen by the equivalence relations.

Consistency of physical meaning can also be established without arbitrary re-interpretation. Accepting Heisenberg's initial classical assumption the Fourier series (11) can be expanded (using conjugate symmetry) as:

$$x(t) = \sum x_\alpha e^{i(\alpha\omega)t} = \sum X_\alpha e^{i(\alpha\omega)t} + \sum \bar{X}_\alpha e^{-i(\alpha\omega)t}$$

$$where\ X_\alpha = \tfrac{1}{2}x_\alpha \quad (44a)$$

$$leading\ to\ [\boldsymbol{q_1}, \boldsymbol{p}_1] - [\boldsymbol{q_2}, \boldsymbol{p_2}] \Rightarrow [\boldsymbol{q_1}, \boldsymbol{p}_1] - [\overline{\boldsymbol{q_1}}, \overline{\boldsymbol{p}_1}] = i\hbar \quad (44b)$$

Using equivalence condition (38) with complex conjugate definitions from (44b), leads directly to relations (40b, c) and (41a, c) and then to the individual non-commutations relations (32). All other results follow. Alternatively, commencing with (44a) and following Born-Jordan leads directly to (32) assuming equivalence. This result is most interesting. Endogenous motion is implicit in the mathematics of standard matrix mechanics.

For the unit transition position and momentum matrices are:

$$\boldsymbol{Q}(n \leftrightarrows m,t) = \begin{bmatrix} 0 & q(n,m,t) \\ q(n,m,t) & 0 \end{bmatrix} \tag{45a}$$

$$\boldsymbol{P}(n \leftrightarrows m,t) = \begin{bmatrix} 0 & p(n,m,t) \\ p(n,m,t) & 0 \end{bmatrix} \tag{45b}$$

$$giving: \ [\boldsymbol{Q},\boldsymbol{P}]_{(n\leftrightarrows m,t)} = 0 \tag{45c}$$

Both are Hermitian and commuting but not in general quaternions, which is different from the treat of the HO. The preliminary HO analysis was aimed at establish a physical basis for the model, and was not fully consistent with matrix mechanics. Unit transition eigenvalues and corresponding eigenstates are:

$$Q_{eigenvalue}^{(n\leftrightarrows m,t)} = (\pm)\sqrt{2}[q_R cos\omega(n,m)t - q_I sin\omega(n,m)t] \tag{46a}$$

$$P_{eigenvalue}^{(n\leftrightarrows m,t)} = (\pm)\sqrt{2}[p_R cos\omega(n,m)t - p_I sin\omega(n,m)t] \tag{46b}$$

$$Q_{qigenstate} = \begin{bmatrix} Q_{eigenvalue}^{(n\leftrightarrows m,t)} \\ q(n,m,t) \end{bmatrix} \tag{46c}$$

$$P_{eigenstate} = \begin{bmatrix} P_{eigenvalue}^{(n\leftrightarrows m,t)} \\ p(n,m,t) \end{bmatrix} \tag{46d}$$

These relations describe an eigenpath. Magnitudes of the eigenvalues are the same as those of the endogenous quantities (41). Because of relation (45c) position and momentum eigenvalues are simultaneously measurable. The significance of these relations have been previously discussed [9]. Although with the preliminary treatment eigenpath position and momentum were not simultaneously measurable, that discussion remains relevant. That position and momentum for the proposed micro eigenpaths are simultaneously measurable is an empirical departure from standard QM.

For the HO, ensemble position and momentum are obtained directly using (41a) and (41c) together with BJ transition amplitudes without further calculations. The energy- matrix using (33b) is diagonal with elements reproducing standard eigenvalues. With the preliminary treatment, the classical Hamilton was incorrectly used and was found to not be diagonal. Uncertainty relations where uncertainty is defined as standard deviations of position and moment trajectory relations (41) are also reproduced. Of interest is minimum uncertainty for the HO ground state whether trajectories are:

$$q(n,n+1,t)_{end} = \left[\left(\frac{\hbar}{\omega}\right)\right]^{½} \cos(\omega t) \tag{47a}$$

$$p(n,n+1,t)_{end} = [(\hbar\omega)]^{½}\sin(\omega t) \tag{47b}$$

$$giving (\Delta q \Delta p)_{min} = ½\hbar \ and \ < H > = ½(\hbar\omega) \tag{47c}$$

Relation (47c) identifying the minimum expectation energy with the HO ground state, is obtained following the QM mathematical reasoning of minimizing the expectation value of the classical Hamilton together with the position-momentum minimum uncertainty expression. While the mathematics is the same the physical basis is different. Uncertainties and minimum energy (47c) are based on the proposed endogenous motions mathematically expressed by (47a, b).

Unit transition matrices of (44) for each individual component term of relations (41a, c) are also modified unit transition amplitudes matrices. Using relations (40b, c) to define the modified transition amplitudes leads to the following relations for the HO as particular case:

$$\left[\boldsymbol{q_{BJ}}^{\boldsymbol{M}}, \boldsymbol{p_{BJ}}^{\boldsymbol{M}}\right]_{(n \leftrightarrows m)} = (n+1)\tfrac{1}{2} i\hbar\sigma_3 \tag{48a}$$

$$where\ \ \boldsymbol{q_{BJ}}^{\boldsymbol{M}} = \sqrt{\tfrac{1}{2}}\boldsymbol{q_{BJ-unit}}\ ;\ \boldsymbol{p_{BJ}}^{\boldsymbol{M}} = \sqrt{\tfrac{1}{2}}\boldsymbol{p_{BJ-unit}} \tag{48b}$$

$$with\ \boldsymbol{q_{BJ}}^{\boldsymbol{M}} = \left[\left(\frac{\hbar}{\omega}\right)(n+1)\right]^{\frac{1}{2}} (\cos(\omega t)\,\tfrac{1}{2}\sigma_1 + \sin(\omega t)\,\tfrac{1}{2}\sigma_2) \tag{48c}$$

$$with\ \boldsymbol{p_{BJ}}^{\boldsymbol{M}} = [(\hbar\omega)(n+1)]^{\frac{1}{2}}(-\sin(\omega t)\,\tfrac{1}{2}\sigma_1 + \cos(\omega t)\,\tfrac{1}{2}\sigma_2) \tag{48d}$$

Relation (48c, d) expresses the modified unit transition position and momentum matrix in quaternion representation in Pauli notation. For the ground state transition the RHS are mathematically equivalent to the corresponding spin components. Assuming this equivalence is physically meaningful, the relations suggest an association between spin and unit transition, at least to the extent of similar mathematical properties.

4. EPR-conundrum

Mermin gives an alternative analysis of EPR by considering a Bell-type experiment [22]. Briefly, singlet spin particles emerge from a source to two spatially separated detectors. Each detector can be randomly set in one of three orientations. Each device can flash green (G) or red (R) depending on spin, where measurement is set such that a particle pair with opposite spin along the same orientation flash the same colour. Mermin considers two cases: case (a) is where each device has the same setting and where both detectors flash the same colour. Case (b) is where detectors have different settings and where both detectors flash the same colour a quarter of the time. Outcomes are predicted by QM.

Mermin introduces the notion of the particle as the carrier of information by way of instruction sets which inform measurement outcomes. Accordingly, each particle can be classified into eight types: {RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG}, for each of three possible outcomes. Assuming both particles carry the same information, then for case (a) detectors will always flash the same colour because particles carry the same instruction set. This outcome agrees with QM. Assuming the same colour code hypothesis for case (b) the same colour occur at least a third of the time, which is contrary to QM (unless RRR and GGG never occur). Clearly, there is a conundrum.

While alternative interpretations have been proposed, it will be accepted that the conundrum is genuine [23].Since instruction sets at set at source, thereby defining initial conditions, local causality is preserved.

Colour combinations assign outcomes on unmeasured orientations assuming pre-existing values. Accordingly, the undefined property hypothesised by Mermin must instruct a classical measurement. However, since QM is agnostic to non-measurement outcomes, the correct instruction set must differ by instructing a non-classical measurement. Basic QM requires that: a) outcomes are random, b) net zero spin conservation, and c) eigenstates are mutually exclusive. It is reasonable to assume that the correct instruction set should adhere to these basics. Assuming the particle is indeed defined by its properties, a basic flaw in Mermin's EPR conundrum is to assume a classical particle.

This conclusion may well appear too elementary! However, it must also be considered whether thought experiments, paradoxes, no-go theorem and conundrums are sufficiently detailed to enable conclusions which are otherwise.

At least for position and momentum, the proposed HV define QM variables not directly the results of QM experiments. However, any HV model (including one for spin) which

reproduces quantum variables would reproduce results of quantum experiments. This raises the question whether violation actually excludes these types of models. Despite claims of its generality, does Bell's hidden variable parameter make assumptions about model-type?

A QM measurement is measurement of the generated eigenstate - eigenvalue system, where the outcome need *not* be pre-determined. If as hypothesised, the QM particle contains all ensemble information then in pre-measurement the quantum object need not be in a specific eigenstate. Accordingly, a pre-measurement eigenvalue cannot be specified. Measurement procedures for hidden and QM variables must then differ, requiring quantum measurements to be apparatus-interactive [9, 3, 24]. That reality exists is never in question [25]. While aspects of Copenhagen-type subjectivism persist, its origin is localised within the dimensions of the QM particle and would be attributed to its internal endogenous properties, presently not well known [26].

Bohmian mechanics (BM) is a widely considered HV theory which has accompanied the foundations question. An exploratory comparison is appropriate. The theory attempts to address conceptual difficulties of QM, such as the measurement problem and iconic double-slit experiment, among others. As such it represents an ontological shift away from the orthodoxy that QM is primarily about measurement to being about particles. That implies standard QM is incomplete. Conceptually, as discussed, the proposal model accords with this view.

There are however differences: BM focuses on the wave function giving an incomplete description of the quantum state, while this proposal works with matrix mechanics. For BM the state of the system is described by the wave function supplemented by particle position evolving according to an additional guiding equation of motion. There is nevertheless commonality in that both approaches introduce position as a fundamental HV describing deterministic motion. Whereas for this model, the motion is endogenous, periodic and Newtonian, Bohmian trajectories are non-Newtonian. While observation of Bohmian position is problematic, for this model position measurability is defined mathematically.

There is a further difference on how wave function incompleteness is defined. For this proposal the HV motion averages out over a cycle; as such it is unseen by the wave function, which remains unaltered as described by the Schrodinger equation. That means the wave function is restricted but complete in what it describes. Bohmian mechanics does not address the question of whether the wave function is itself a consequence of latent deeper structure. Since the mathematics of BM equations of motion is not impacted by this proposal, the two approaches are not *necessarily* in contradiction but can compliment. Collapse of the wave function and double slit experiment explanations of BM are not affected. Nevertheless, the proposed model does allow for an alternative interpretation to non-locality for interdependence of particle positions. An ensemble of quantum particles moving along deterministic endogenous paths, and subject to conservation constraints, *may* be explained by synchronicity rather than non-locality.

5. Conclusion

This work follows a different path in exploring the foundation question by proposing an *actual* hidden variables model. As a more traditional starting point, the physical assumptions and mathematical consequences of Heisenberg and also Born in formulating matrix mechanics are re-visited. Bohr's instantaneous state-to-state transition has been experimentally invalidated, and is rejected. Heisenberg's non-path postulate is replaced by periodic endogenous paths during the now finite transition. Although in disagreement with some interpretations of QM and Bell inequalities violations, the model is nevertheless guided

by EPR-Bell analysis. The resulting modified matrix mechanics is mathematically equivalent to Born-Jordan reproducing all standard results. Indeed, endogenous motion is found to already be implicit in the mathematics of standard matrix mechanics. Because such motion averages out over a cycle it is unseen by the wave function. Nevertheless, mathematical equivalence with position and momentum non-commutation of the Schrodinger equation is preserved.

The proposed model does raise questions relevant to EPR-Bell: whether a re-definition of wave function completeness is required, and whether additional HV model-type assumptions are implicit in Bell's hidden variables parameter.

Replacing the non-path postulate is not however a claim that QM is not a departure from classicality: an assumption of the model is that a quantum particle is not classical.

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