Original Paper

The Kinematic Structure of the Relativistic Quantum Mechanics Equations and What May Lie Behind It

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Abstract: The concept of kinematics is adopted to find the kinematic forms of relativistic quantum mechanics (Dirac wave function, Dirac equation and Klein-Gordon equation). The kinematic forms do not contain Planck's constant but have the imaginary i. Since 2007, a theory has been developing to explain the complexity (imaginary i) which is a result of the physical process. It may be referred to as the "circles theory". This theory showed a complex position vector, a complex velocity, and a complex acceleration for a point in a system of two rolling circles. Interestingly, these three equations are similar to the kinematical forms of relativistic quantum mechanics.

Keywords: Planck constant, wave function, physical complexification, partial observation, imaginary i, circles theory

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1. Introduction

The quantum mechanics theory developed in different stages from 1900. The first stage was Planck's proposal of the energy quantisation concept (Planck, 1901). In this stage, Planck's constant (h) was introduced via energy quantisation:

$$E = hf \tag{1}$$

where E is the energy and f is the frequency of the spectrum.

During 1925 and 1926, Heisenberg, Born, Jordan, and Schrödinger started to formulate the dynamical forms of the theory (Heisenberg 1925; Born 1925; Born, et al., 1926; Schrödinger 1926). These were in matrix forms or wave equation form. In 1928, Dirac proposed his equation of relativistic quantum mechanics (Dirac, 1928). However, the common feature in all of these forms is the imaginary i.

The synthesis of these and other achievements was realized by John von Neumann in 1932 (von Neumann, 1932). The achievements were formulated in an axiomatic form to constitute an axiomatic theory and to constitute the mathematical foundation of the quantum mechanics theory.

One may observe that there are two distinguished features in mathematical formulations of quantum mechanics, which are quite different from the classical or macroscopic physics. These are:

1- The quantization by quantity $\hbar$ which is the Planck constant. It is the physical quantity and its unit is related to the energy (J) and the time (s).

2- The imaginary i that makes the mathematical complexity feature of the quantum mechanics mathematical expirations.

1.1 The kinematics

The physical theories, in general, may have two different forms of mathematical representations (equations). These are the dynamical equations and the kinematical equations (Curiel, 2016). The kinematics may be defined as "the study of the motions of particles and rigid bodies, disregarding the forces associated with those motions. It is purely mathematical in nature and does not involve any physical laws such as Newton's laws" (Greenwood, 1965).

For a physics theory, kinematics may refer to the *geometry of motion*. In this regard, the mathematical space (classical or relativistic) is the ground play of motion in the theory but without forces. The basic kinetic equations of motion started from the determination of position with time toward the acceleration (Fig. 1).

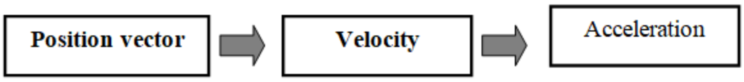

Fig. 1 The kinematics equations

Therefore, the analysis of a possible space-time model (or kinematics) has a fundamental meaning for physics (Gromov, Kuratov, 2004).

Within quantum mechanics, there have been a few attempts to derive the quantum mechanics equations without Planck's constant. These attempts propose new forms of equations (Barut, 1992 and Ralston, 2012). Barut tried to formulate quantum mechanics without the parameters Planck's constant , mass and charge as a pure "wave theory" in terms of the frequencies alone. "This is more directly related to experiments where one measures frequency differences rather than energies. Different quantum systems are then characterized by an intrinsic proper frequency $\omega_\circ$ "(Barut, 1992). In Barut's attempt, his derivation for the Dirac equation (for free particle) is:

$$i\frac{\partial\psi}{\partial(ct)} = \left[-i\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta\left(\frac{\Omega}{c}\right)\right]\psi. \qquad (2)$$

"In the rest frame of the localized lumps the frequency is Ω which gives the system the inertia (mass) as it appears in the dispersion relation" (Barut, 1990).

1.2. Aim of the work

In the present article, we try to find kinematical forms for the relativistic quantum mechanics equations with respect to the quantum mechanics equation forms and do not try to find another version. Then, we compare them with a previous kinematic and non-quantum mechanics work, showing an analogy with relativistic quantum mechanics equations. That work may be termed as "circles theory".

2. Quantum mechanics equations

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Relativistic quantum mechanics presents a more general form of quantum mechanics. Furthermore, relativistic quantum mechanics is based on a combination of two independent theories, special relativity and quantum mechanics. Thus, this work will deal with relativistic quantum mechanics.

The Klein-Gordon equation (Klein, 1926; Gordon, 1926) is a combination of the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi\,, \tag{3}$$

and the relativistic Hamiltonian of a free particle:

$$H^2 = c^2P^2 + m^2c^4. \tag{4}$$

The Klein-Gordon equation is a second-order time-differential equation,

$$i^2\hbar^2\frac{\partial^2\psi}{\partial t^2} = [c^2P^2 + m^2c^4]\psi \tag{5}$$

whereas Dirac's equation (Dirac, 1928) is a first-order time-differential equation,

$$i\hbar\frac{\partial\psi}{\partial t} = (c\boldsymbol{\alpha}\cdot\boldsymbol{P} + \beta mc^2)\psi \tag{6}$$

Both Hamiltonians (the squared $(c^2P^2 + m^2c^4)$ and the linearized $(c\boldsymbol{\alpha}\cdot\boldsymbol{P} + \beta mc^2)$) are inserted into the quantum formulation, or it is an artificial technique to merge both theories. In other words, there is no theory that can show a complete derivation without merging two different theories. Relativistic quantum mechanics is emergent of two different theories, special relativity and quantum mechanics.

However, the trial solution of the Dirac equation consists of a combination of a spinor $[u_D(x,t)]$ and a complex phase factor. The two factors (spinor and complex phase) have different conceptual roots, but they form a mathematical structure for the solution (ψ_D). For a free particle, the trial solution has the form:

$$\psi_D(x,t) = u_D(x,t)\exp\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x} - Et), \tag{7}$$

where, ω, k, and $u_D(x,t)$ are angular frequency, wave vector, and a Dirac four-component spinor respectively. The structure of the spinor depends on the nature of the Dirac Hamiltonian for the studied case.

3. Formulation without Planck's constant

In considering the de Broglie, Planck, and Einstein equations:

$$\hat{p} = -i\hbar\nabla\,, E = \omega\,\hbar \text{ and } \omega = m\,c^2/\hbar, \tag{8}$$

The angular frequency of these equations (ω) is related to the energy of rest mass frequency. We can say that Eqs. (5, 6) contain the constant $\hbar$ on both sides of the equations. At the same time, these equations contain the imaginary i but on one side of the equations. The complex phase factor of Eq. (8) is multiplied and divided by $\hbar$ and multiplied by the imaginary i. Accordingly, one can cancel Planck's constant from each side without any mathematical problem. Then, rewrite these equations without $\hbar$. The Klein-Gordon (Eq. (5)) becomes:

$$i^2\frac{\partial^2\psi}{\partial t^2} = [-c^2(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}) + \omega^2]\psi\,. \tag{9}$$

Dirac equation (Eq. 6) becomes:

$$i\frac{\partial\psi}{\partial t} = (-ic\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta\omega)\psi\,, \tag{10}$$

and Dirac wave function (Eq.7) becomes:

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$$\psi_D = u_D \exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t). \tag{11}$$

$\boldsymbol{k}$ is the wave number of de Broglie wave.

It is evident that the units of Eqs. (9, 10, 11) are for space and time and fulfil the above definition of kinematics. Then, Eqs. (9, 10, 11) are kinematical forms of Klein-Gordon equation, Dirac equation, and Dirac's wave function respectively. Even though Planck's constant is not present, these equations are still equations of quantum mechanics.

Dirac form (Eq. (10)) is similar to Barut's form (Eq.(2)). These formulas are not related to a certain kinematical model as in the Barut's attempt. Now, the question is, what type of kinematical system are these equations related to? Are they related to those mentioned in Fig. 1? If the answer is yes, so what is the system?

4. Space-time structure and the imaginary i

Before trying to get answers to the above questions, we must observe that it is obvious that the kinematic forms (Eqs. (9, 10, 11)) are associated with the imaginary i, and according to these kinetic forms, many more questions may arise, such as:

- Does it mean that the space-time structure of the relativistic quantum mechanics as well as quantum mechanics is complex?
- In that case, there is a relationship between quantum and special relativity.
- Then, is the complexity associated with the microscopic world?

Since 2007, a mathematical model is being developed to study the complexity (appearing of imaginary i in physics equations) as a cause of physical causes (Sanduk, 2007, 2012, 2008a, 2008b, 2019,…). This work has no relationship with quantum mechanics. It is a kinematical theory based on a study of the motion of rolling circles. Thus, as mentioned before, it may be termed as "circles theory". The timeline of the circles theory has been shown in Fig. 2.

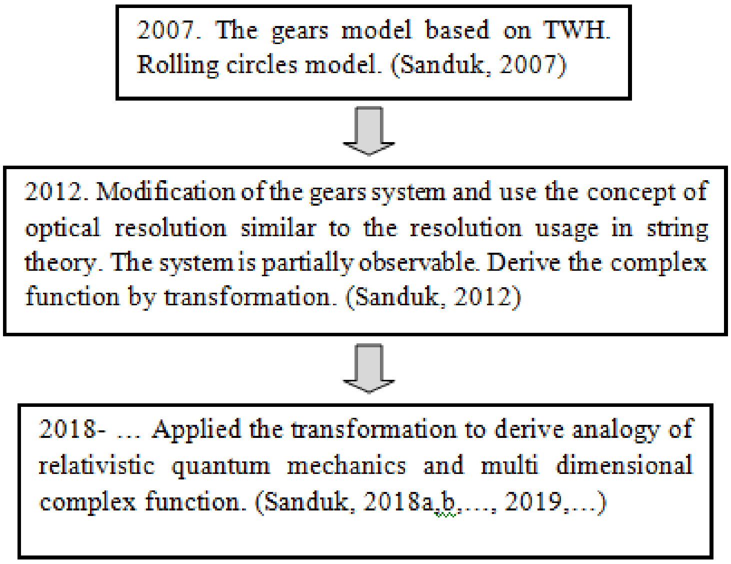

Fig.2. The timeline of the circles theory

4.1. Circles theory

The theory is based on two postulates. The first is related to a system in the external world, and the second is related to the observation problem.

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The first postulate is a proposal of the external world system. It is a mathematical model of two rolling circles in a real plane as shown in Fig.3. The second postulate is related to the observation of the system. The system may be perceived in different forms according to the properties of used light, which is the data carrier from the object to the lab observer (Sanduk, 2012, 2018a, 2018b). The problem of optical resolution may lead to missing data or a problem of partial observation. The observation is introduced to play a serious rule in forming the representation of the external world by the lab observer. It is worth mentioning that the string theory has used the concept of resolution to explain the point particle. But, in circles theory, this problem causes big changes. It leads to explaining the point particle and the associated wave. In addition to that, it leads to complex form or physical complexification (Sanduk, 2019a).

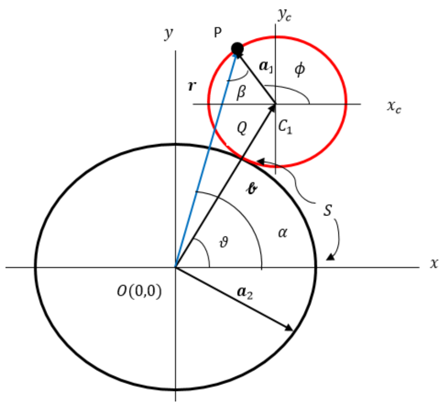

Fig.3. The rolling circles systems (Sanduk, 2018a)

The theory studies the kinematics of a point (P) as in Fig.3, the position vector, the velocity and the acceleration equations. Then, these equations are transformed under the effect of partial observation. The results show:

- The complex position vector (Sanduk, 2018a):

$$\boldsymbol{\mathcal{Z}}(\boldsymbol{s},t) = \boldsymbol{a}_{2m} \exp \pm i(\boldsymbol{k}_{2m} \cdot \boldsymbol{s} - \omega_{1m} t). \tag{12}$$

The subscript m indicates the resolved (measured) values.

- The complex velocity (Sanduk, 2018b):

$$i\frac{\partial \boldsymbol{\mathcal{Z}}}{\partial t} = (-iv\boldsymbol{A} \cdot \nabla + B\omega_{1m})\boldsymbol{\mathcal{Z}}\,, \tag{13}$$

where $\boldsymbol{A}$ and B are the coefficients related to the rotation of the system.

- The complex acceleration equations (Sanduk, 2018b):

$$i^2\frac{\partial^2 \boldsymbol{\mathcal{Z}}}{\partial t^2} = (-v^2\nabla^2 + {\omega_{1m}}^2)\boldsymbol{\mathcal{Z}}\,. \tag{14}$$

The Eqs. (12, 13, 14) are kinematical equations with imaginary i. The imaginary i arises owing to the problem of observation. They are not related to quantum mechanics.

4.2. Comparison with relativistic quantum mechanics

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As mentioned above, the circles theory is not a quantum mechanics theory and certainly does not take into account the Planck constant. The theory shows complex kinetic equation forms. In order to show a relationship between the circles theory equations (Eqs. (12, 13, 14)) and the kinematic equation of the relativistic quantum mechanics (Eqs. (9, 10, 11)), both groups have been presented in a table. Table 1 shows the comparisons between the forms of the kinematic relativistic quantum mechanics (Eqs. (9, 10, 11)) and the forms obtained by the circles theory (Sanduk, 2019a). It is obvious that the agreement is quite good.

Table 1 The comparisons (Sanduk 2018a, 2018b).

Conventional definition	Kinematic equations of relativistic quantum mechanics	Forms of the circles theory	Definition
Dirac wave function	$\psi_D = u_D \exp i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)$	$\boldsymbol{\mathcal{Z}} = \boldsymbol{a}_{2m} \exp \pm i(\boldsymbol{k}_{2m} \cdot \boldsymbol{s} - \omega_{1m} t)$	*Z-complex vector*
Dirac equation	$i\frac{\partial\psi}{\partial t} = (-ic\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta\omega)\psi$	$i\frac{\partial\boldsymbol{\mathcal{Z}}}{\partial t} = (-iv\boldsymbol{A}\cdot\boldsymbol{\nabla} + B\omega_{1m})\boldsymbol{\mathcal{Z}}$	*Complex velocity equation*
The coefficients	$\boldsymbol{\alpha}$ and β	$\boldsymbol{A}$ and B	*Coefficients*
Property	$\boldsymbol{\alpha}_i\boldsymbol{\alpha}_j + \boldsymbol{\alpha}_j\boldsymbol{\alpha}_i = 0$	$\boldsymbol{A}_\theta \cdot \boldsymbol{A}_\varphi + \boldsymbol{A}_\varphi \cdot \boldsymbol{A}_\theta = 0$	*Property*
Property	$\boldsymbol{\alpha}_i\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_i\boldsymbol{\alpha}_i = 2$	$\boldsymbol{A}_\theta \cdot \boldsymbol{A}_\theta + \boldsymbol{A}_\theta \cdot \boldsymbol{A}_\theta = 2$	*Property*
Property	$\alpha_i^2 = \beta^2 = 1$	$A^2 = B^2 = 1$	*Property*
Property	$\boldsymbol{\alpha}_i\beta + \beta\boldsymbol{\alpha}_i = 0$	$\boldsymbol{A}B + B\boldsymbol{A} = 0$	*Property*
Klein-Gordon equation	$\frac{\partial^2\psi}{\partial t^2} = [c^2\nabla^2 - \omega^2]\psi$	$\frac{\partial^2\boldsymbol{\mathcal{Z}}}{\partial t^2} = [v^2\nabla^2 - {\omega_{1m}}^2]\boldsymbol{\mathcal{Z}}$	*Complex acceleration equation*

5. Notes

- Kinematics quantum mechanics equations are associated with the imaginary i. So, imaginary i is related to the space-time structure.
- According to the circles theory, the imaginary i is related to the observation problem in the microscopic world.
- The circles theory may throw light on a foundation of the axiom of quantum mechanics (Sanduk, 2019b).
- Quantum and special relativity both may be related to the same origin.
- The equations of the circles theory are similar to the kinematic form of the relativistic quantum mechanics equations.
- Circles theory does not accept the concept of Planck’s length. Still, one may ask, are these circles are related to the concept of granular space and time (The space-time arises from the circles combination and the problem of observation)?
- Another type of physics may be existed behind quantum physics!

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References

Barut, A. (1990). $E = \hbar\omega$, Phys. Lett. A143, 349.

Barut, A. (1992). Formulation of wave mechanics without the Planck constant h, Physics Letters A. 171, Issues 1–2, 30 November 1992, pp. 1–2.

Born, M. and Jordan, P. (1925). Zur Quantenmechanik, Z. Phys., 34, pp 858–888.

Born, M., Heisenberg, W., and Jordan, P., 1926, Zur Quantenmechanik II, Z. Phys., 35, pp 557–615.

Curiel, E. (2016). Kinematics, Dynamics, and the Structure of Physical Theory, arXiv:1603.02999v1 [physics.hist-ph] 9 Mar 2016.

Dirac, P. (1928). The quantum theory of the electron, Proceeding of the Royal Society A: Mathematical, Physical and Engineering Sciences, 117 (778), pp. 610–624.

Gordon, W. (1926). Der Comptoneffekt nach der Schrödingerschen Theorie, Z. Phys. 40, 117.

Gromov, N., Kuratov, V. (2004). Quantum kinematics, arXiv:hep-th/0410086v1, 8 October 2004.

Greenwood, D. (1965,). Principles of Dynamics, Prentice-Hall, Englewood Cliffs, N.J.

Heisenberg, W. (1925). Uber Quantentheoretische Umdeutung Kinematischer und Mechanischer Beziehungen, Z. Phys., 33, pp. 879–893.

Klein,O. (1926). Quantentheorie und fünfdimensionale Relativitätstheorie, Z. Phys., 37, 895.

Planck, M. (1901). Ueber das Gesetz der Energieverteilung im Normalspectrum, Ann. Phys., 309 (3): 553–63, Biccode: 1901 AnP…309..553P.

Ralston, J. (2012). Quantum Theory without Planck's Constant, arXiv:1203.5557v1 [hep-ph] 25 Mar 2012.

Sanduk, M. (2007). Does the Three Wave Hypothesis Imply a Hidden Structure? Apeiron, 14, No. 2, pp. 113–125.

Sanduk, M. (2012). A kinematic model for a partially resolved dynamical system in a Euclidean plane, Journal of Mathematical Modelling and Application, **1**, No. 6, pp. 40–51. ISSN: 2178–2423.

Sanduk, M. (2018a). An Analogy for the Relativistic Quantum Mechanics via a Model of de Broglie Wave-Covariant æther, International Journal of Quantum Foundations, 4, pp. 173–198.

Sanduk, M. (2018b). From a Classical Model to an Analogy of the Relativistic Quantum Mechanics Forms – II, International Journal of Quantum Foundations, 4, pp. 223–234.

Sanduk, M. (2019a). Is there a physical reason beyond the imaginary *i* in the quantum mechanics formulation? International Journal of Quantum Foundations, 5, pp. 69–79.

Sanduk, M. (2019b). Is There a Physical Reality Beyond the Axioms of Quantum Mechanics? Quantum Speculations **1**, pp. 85–92.

Schrödinger, E. (1926). Quantisierung als Eigenwertproblem (Vierte Mitteilung), Annalen der Physik, 81, pp. 109–139.

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von Neumann, J. (1932). Matematische Grundlagen der Quantenmechanik, Springer, Berlin.