

*Original Paper*

## **Is there a physical reason beyond the imaginary $i$ in the quantum mechanics formulation?**

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**Abstract:** The imaginary  $i$  in the formulation of the quantum mechanics is accepted within the axioms of the quantum mechanics theory, and, thus, there is no need for an explanation of its origin. Since 2012, in a non-quantum mechanics project, there has been an attempt to complexify a real function and build an analogy for relativistic quantum mechanics. In that theoretical attempt, a partial observation technique is proposed as one of the reasons behind the appearance of the imaginary  $i$ . The present article throws light on that attempt of complexification and tries to explain the logic of physics behind the complex phase factor. This physical process of partial observation acts as a process of physicalization of a virtual model. According to the positive results of analogy, the appeared imaginary  $i$  in quantum mechanics formulation *may be* related to a partial observation case as well.

**Keywords:** Complex phase factor, wave function, physical complexification, partial observation, physicalization, imaginary  $i$ .

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### **1. Introduction**

The dealings with the complex mathematical form is well known in classical physics. For example, the main use of the complex traveling wave in classical physics and engineering is to simplify the calculations using sinusoidal varying quantities. In these applications, the physical quantities are obtained by considering the real part, whereas the imaginary part is left out. In those applications, the complex traveling waveform is merely an optional mathematical technique.

The case is quite different in quantum mechanics. In 1925, Heisenberg, Born, and Jordan proposed the matrix mechanics (Heisenberg 1925; Born 1925; Born 1926). The imaginary  $i$  entered quantum physics in 1925 through these works. Heisenberg (1925) introduced the complexity by saying:

*“Along with the frequencies, the amplitudes are necessary for the description of radiation; **the amplitudes can be regarded as complex vectors** (with six independent determining data), and they determine polarization and phase.”*

Then, he started with the real part of the complex number as in classical mechanics.

In his search for an equation that describes the de Broglie waves, Schrödinger derived his equation in 1926 (Schrödinger 1926a, b, c, d, e, f). This attempt has led to what is known as wave mechanics. This is the second formulation for quantum mechanics, where the other formulation is matrix mechanics. However, the time-dependent Schrödinger equation is a second order differential equation with imaginary  $i$ :

$$i\hbar \frac{\partial \psi_S}{\partial t} = \hat{H} \frac{\partial^2 \psi_S}{\partial x^2}, \quad (1)$$

where,  $\psi_S$ ,  $\hat{H}$ , and  $\hbar$  are the Schrödinger's wave function, Hamiltonian, and reduced Planck constant, respectively. The solution of this equation for a free particle may have this form:

$$\psi_S = A \exp i(k \cdot x - \omega t), \quad (2)$$

where,  $\omega$ ,  $k$ , and  $A$  are angular frequency, wave vector, and amplitude, respectively. This is the Schrödinger's wave function. The imaginary  $i$  appeared in Schrödinger formulation again. In his work, Schrödinger showed that the wave function is a product of an amplitude factor and a complex phase factor (Moore 1993). For Schrödinger, the wave function represents the maximal possible knowledge:

*"Maximal knowledge of a total system does not necessarily include total knowledge of all its parts, not even when these are fully separated from each other and at the moment are not influencing each other at all"* (Schrödinger, 1935).

The Schrödinger's wave equation was a non-relativistic form. In 1928, Dirac proposed the relativistic form:

$$i\hbar \frac{\partial \psi_D}{\partial t} = (c\alpha \cdot \hat{p} + \beta mc^2)\psi_D, \quad (3)$$

where,  $\beta$  and  $\alpha$  are the Dirac matrices,  $\hat{p}$  is the momentum operator,  $m$  is the rest mass of the particle,  $\hbar$  is the reduced Planck constant (Dirac constant),  $\psi_D$  is the Dirac wave function, and  $c$  is the light velocity. This relativistic form demonstrates the imaginary  $i$  as well.

The trial solution of the Dirac equation consists of a combination of a spinor [ $u_D(x, t)$ ] and a complex phase factor. The two factors (spinor and complex phase) have different conceptual roots, but they form a mathematical structure for the solution ( $\psi_D$ ). For a free particle, the trial solution has the form:

$$\psi_D(x, t) = u_D(x, t) \exp i(k \cdot x - \omega t), \quad (4)$$

where,  $\omega$ ,  $k$ , and  $u_D(x, t)$  are angular frequency, wave vector, and a Dirac four-component spinor, respectively. The structure of the spinor depends on the nature of the Dirac Hamiltonian for the studied case. Furthermore, The imaginary  $i$  appears in the Dirac formulation again.

The complex phase factor:

$$\text{Complex phase factor} \equiv \exp i(k \cdot x - \omega t), \quad (5)$$

appears in both forms for the wave functions (nonrelativistic equation (3) and the relativistic equation (5)).

Dirac described the complex phase *"...this phase is all important because it is the source of all interference phenomena, but its physical significance is obscure. So the real genius of Heisenberg and Schrödinger, you might say, was to discover the existence of probability amplitudes containing this phase quantity which is very well **hidden** in nature and*

it is because it was so well hidden that people hadn't thought of quantum mechanics much earlier" (Dirac 1972; Yang, 1989).

### 1.1 The origin of the imaginary $i$

The proposers of both formulations (Matrix and wave mechanics) have not shown a clear justification for the appearance of the imaginary  $i$  in their works. Thus, there are many contributions from historians of science, physicists, and mathematicians who have tried to provide explanations for it (Baylis 1992; Poojary 2014). As an example, the interesting explanation by the theoretical physicist Freeman Dyson in his article "Birds and Frogs" mention the technique that has been used by Schrödinger:

*"One of the most profound jokes of nature is the square root of minus one that the physicist Erwin Schrödinger put into his wave equation when he invented wave mechanics in 1926. Schrödinger was a bird who started from the idea of unifying mechanics with optics. A hundred years earlier, Hamilton had unified classical mechanics with ray optics, using the same mathematics to describe optical rays and classical particle trajectories. Schrödinger's idea was to extend this unification to wave optics and wave mechanics. Wave optics already existed, but wave mechanics did not. Schrödinger had to invent wave mechanics to complete the unification. Starting from wave optics as a model, he wrote down a differential equation for a mechanical particle, but the equation made no sense. The equation looked like the equation of conduction of heat in a continuous medium. Heat conduction has no visible relevance to particle mechanics. Schrödinger's idea seemed to be going nowhere. But then came the surprise. Schrödinger put the square root of minus one into the equation, and suddenly it made sense. Suddenly it became a wave equation instead of a heat conduction equation. And Schrödinger found to his delight that the equation has solutions corresponding to the quantized orbits in the Bohr model of the atom. It turns out that the Schrödinger equation describes correctly everything we know about the behavior of atoms. It is the basis of all of chemistry and most of physics. And that square root of minus one means that nature works with complex numbers and not with real numbers. This discovery came as a complete surprise, to Schrödinger as well as to everybody else."* (Dyson 2009).

However, this complex formulation is not that the one that is used in different applications in physics, but it is necessary for quantum mechanics formulation (Wigner 1960): *"Surely to the unpreoccupied mind, complex numbers are far from natural or simple and they cannot be suggested by physical observations. Furthermore, the use of complex numbers is in this case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics."*

Some physicists have stated the possibility of formulation in quantum mechanics without the imaginary  $i$  and regard that a proof of that complex formulation is not more than a mathematical technique. t' Hooft, in his theory of underlying deterministic quantum mechanics, mentioned that:

*"Some critical readers were wondering where the complex numbers in quantum mechanics should come from, given the fact that we start off from classical theories. The answer is simple: complex numbers are nothing but man-made inventions, just as real numbers are. In Hilbert space, they are useful tools whenever we discuss something that is conserved in time (such as baryon number), and when we want to diagonalize a Hamiltonian. Note that*

*quantum mechanics can be formulated without complex numbers if we accept that the Hamiltonian is an anti-symmetric matrix. But then, its eigen values are imaginary.*

*We emphasize that imaginary numbers are primarily used to do mathematics, and for that reason, they are indispensable for physics.” (t’ Hooft 2016).*

Therefore, according to him, there is no need to find where the complex number in quantum mechanics should come from. This is because it is not more than a useful tool but has shown that the imaginary  $i$  cannot be avoided (at the end of his statement, t’ Hooft said, “But then, its eigen values are imaginary”).

During the 1990s, Hestenes (1966, 1990a, b), with respect to geometric algebra, proposed many new concepts related to the complexity of the wave function, such as:

- The imaginary  $i$  can be interpreted as a representation of the electron spin.
- The complex phase factor literally represents a *physical* rotation, the *zitterbewegung* rotation.
- The complex phase factor is the main feature, which the Dirac wave function shares with its nonrelativistic limit. The Schrödinger wave function inherits the relativistic nature.

The vital concept presenting in Hestenes’ proposal is the kinematic origin of the complex phase factor and the physical rotation (*Zitterbewegung*). However, there is no experimental evidence to support the idea of the kinematic origin.

## **2. The necessity of $i$ explanation**

Quantum mechanics is an axiomatic theory that is based on a number of postulates. The postulates form the mathematical foundation of quantum mechanics theory. Historically, John von Neumann mentions these postulates first in 1932 (von Neumann 1955). Moreover, these postulates have not been derived from simpler previously accepted statements. The list of basic axioms of quantum mechanics, as formulated by von Neumann, includes only general mathematical formalism of the Hilbert space and its statistical interpretation.

There is no unanimous agreement on the set of the quantum mechanics postulates. In this article, we are interested in one of the postulates that is related to the wave function (Nottale and C’el’erier 2007): *Complex state function* ( $\psi$ ). Each physical system is described by a state function, which determines all that can be known about the system. The wave function in this postulate is not described as a complex function (Hilbert space), but, due to multiplication with its conjugate for probability calculation (Born rule), the wave function is a complex function.

Owing to this axiomatic nature of the wave function (*Complex state function*), quantum physicists (mean stream and others) work with a complex space while classical physicists work in real space. It seems that each world (microscopic and macroscopic) has its own space. The complex space (Hilbert space) is for microscopic physics, whereas the real space is for macroscopic physics. Thus, for the quantum physicist, their complex space is acceptable as a normal space. In other words, there is no need to explain the origin of the imaginary  $i$  as long as the physical concept behind it is present. Therefore, the problem is not with what the theory of quantum mechanics involves but with what it leaves *out*, namely, *an adequate ontology of*

*structures on space changing with time* (Goldstein 2015). Thus, there is no need to find what that imaginary  $i$  is.

In quantum physics, regarding this complex space, there are huge controversies about the interpretations of the complex wave, complex spin, multi-dimensional wave function, etc. Knowing the origin of the complexity of the wave function or the complex phase factor may lead to an explanation of what the quantum theory leaves out, *namely an adequate ontology of structures on space changing with time*. Here, one may raise a question: Is there a physics beyond the appearance of  $i$  in quantum mechanics? If there is, can that offer interpretation for what the quantum mechanics leaves out? Unfortunately, this question is not desirable by quantum physicists. *However, there is no logical barrier to prevent one from asking about the origin of imaginary  $i$* . We think that  $i$  is not just related to a mathematical formulation, but there may be physics behind it. If so, the explanation for imaginary  $i$  may:

- *lead to form a unified foundation of the theory,*
- *remove the ambiguity of the complex space,*
- *show the relationship between quantum axioms, and*
- *show the relationship between quantum mechanics and the special relativity.*

### 2.1 Is there a way to explain $i$ ?

Is there any physical interpretation of the existence of imaginary  $i$ ? To answer this question, one may ask as well, is there any process for the complexification to transform the real function?

In an interesting approach, Gao have derived the free Schrödinger equation based on an analysis of spacetime translation invariance and relativistic invariance. This “new analysis may not only make the Schrödinger equation in quantum mechanics more logical and understandable, but also help *understand the origin of the complex and multi-dimensional wave function.*” (Gao, 2015).

In 2012, a theoretical attempt was made to study the possibility of real vector complexification. On the basis of that attempt, an analogy for the relativistic quantum mechanics foundation had been obtained. In this article, we have tried to explain a possible physical origin of the imaginary  $i$ .

### 3. The complexification

In some mathematical applications such as equation solving, the complexification techniques for real vector space may be needed. The complexification techniques are based on a mathematical operation of the real vector with the complex numbers (Halmos 1958, 1974). The complexification of a real vector space ( $V$ ) is a tensor product of  $V$  with the complex numbers ( $C$ ). Then, any vector  $v$  in  $V^C$  space becomes:

$$v = v_1 \otimes 1 + v_2 \otimes i, \quad (6)$$

This approach of complexification is a pure mathematical technique, and there is no physical concept behind it. Do any physical process lead to complexification?

The three-wave hypothesis (TWH) had been proposed at the end of the 1970s and during the 1980s by Kostro (1978) and Horodecki (1981, 1982, 1983a, b). The concept of TWH

was represented as a kinematical system of two perpendicular rolling circles in 2007 (Sanduk 2007).

The circle's rotation can simulate the simple harmonic oscillation. The two perpendicular circles (Sanduk 2007) is modified to be two rolling circles in a plane (Fig.1). This model can simulate harmonic oscillation and can provide a complex form, if some of the parameters are eliminated. This elimination process was defined as a partial observation (Sanduk 2012). This approach of the partial observation was developed and used to derive analogies for the Dirac equation and the Klein-Gordon equation (Sanduk 2018a, c). A system of three rolling circles under the partial observation shows a multi-dimensional complex function, which may explain the entanglement concept (Sanduk 2018a).

However, this technique of complexification is a transformation of a kinematical system in real space to a system in complex space and is based on a physical process called the partial observation.

### 3.1 The partial observation technique

This physical complexification approach is neither in quantum mechanics theory nor in quantum mathematical techniques, but it tries to build an analogy for the relativistic quantum mechanics to prove the concept of the physical complexification.

This kinematical model (Sanduk 2007) has been improved as a two rolling circles system (gear) in a real plane, as shown in Fig. 1.

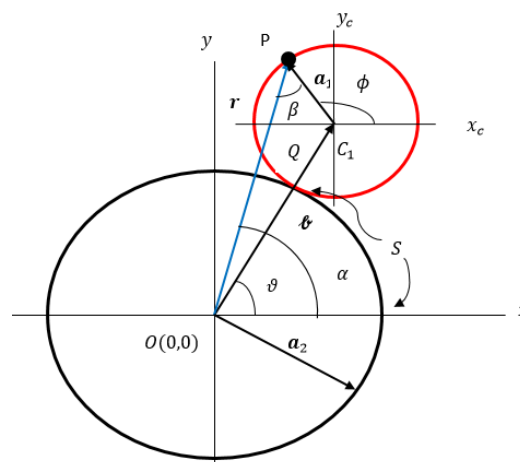


Fig. 1. The real model. Rolling circles model (Sanduk, 2018 a, b, c).

In this system (Fig.1), the position vector ( $r$ ) of a point ( $p$ ) on the first circle is:

$$r = b \left\{ \cos(\vartheta - \phi + \beta) \pm \sqrt{-\sin^2(\vartheta - \phi + \beta) + \left(\frac{a_1}{b}\right)^2} \right\}. \quad (7)$$

From a physical point of view, any object to be observed (lab observation) must be resolved optically. The lab observation is based on the optical spatial resolution (the Rayleigh criterion). For example, for high resolution, the monochromatic light being used should have a wavelength ( $\lambda$ ) smaller than the object dimension ( $x$ ) or whose spatial resolution  $d_\lambda$  is:

$$d_\lambda \ll x, \quad (8)$$

Thus, for any law of physics, all mentioned quantities must be observable or measurable. For example, the length contraction law of the special relativity is:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

All the mentioned quantities must be measurable or observable by the lab observer and satisfy the above condition.

However, the kinematical model (Fig.1) is virtual or in mathematical space and time. To be a physical model, the used light should satisfy the above condition. Then, equation (7) is applicable as it is, where all the quantities can be measured. This condition occurs when the system is in the macroscopic scale. In another case, the model is comparable with the used wavelength. Here, one may get these conditions:

$$a_1 \ll d_\lambda \ll a_2, \quad (9a)$$

and:

$$\omega_1 \gg \omega_\lambda \gg \omega_2. \quad (9b)$$

The lab observer is supposed to deal with the partial observation of the system, where one may say that  $a_1 = \omega_2 = 0$  (unresolved or unobservable quantities), where the  $a_2$  and  $\omega_1$  are well resolved or measured. The physics behind this problem is the use of monochromatic light, which can be applied for the observation of two variables (space and frequency) only, whereas the kinetic model has four variables—two for space ( $a_1, a_2$ ) and two for frequency ( $\omega_1, \omega_2$ ). Therefore, one pair cannot be measured ( $a_1, \omega_2$ ), whereas the other pair can be measured ( $a_2, \omega_1$ ). This is the partial observation.

Due to these conditions, not all the parameters of equation (7) are observable. Then, it may be transformed to (Sanduk 2012, 2018a):

$$Z(s, t, 0) = a_2 \exp \pm i(k_2 \cdot s - \omega_1 t). \quad (10)$$

In this equation, the appearance of the imaginary  $i$  is obvious. It is similar to that of equation (5). The partial observation technique is a transformation of the system in the real plane to a system in the complex plane. This technique has been applied to derive an analogous equation for the Dirac equation and the Klein-Gordon equation (Sanduk 2018c).

#### 4. Comments

- In the physical approach of complexification, the process is an observation process. It makes the system (virtual mathematical model without the condition of observation) as a physical system (under the conditions of observation). It is a process of *physicalization* or to make the virtual system a physical system and can be observed under lab conditions. Therefore, the physical complexification arises due to the physicalization process.

- The complex space arises due to the partial observation problem. Thus, the complex number in quantum mechanics may be related to a physical case and is not just a mathematical formulation.
- The explanation of the physical complexity of the complex phase factor is necessary to elucidate the high dimensional complex function (Sanduk 2018a) and may also provide a reason for the entanglement.
- In order to show a relationship of the two rolling circles model under the effect of the partial observation technique with the relativistic quantum mechanics, this model was used to show an analogy with the relativistic quantum mechanics (Sanduk, 2018 a, b, c). Table 1 shows the comparisons between the forms of the relativistic quantum mechanics and the obtained forms by the physical complexification technique. It is obvious that the agreement is quite good. These comparisons may support the concept of physical complexification.

Table 1. The comparisons (Sanduk 2018a, 2018c).

Conventional definition	Conventional equations of the relativistic quantum mechanics	Analogical model forms	Analogical definition
<b>Dirac wave function</b>	$\psi_D = u_D \exp i(k \cdot x - \omega t)$	$Z = a_{2m} \exp \pm i(k_{2m} \cdot s - \omega_{1m} t)$	<i>Z-complex vector</i>
<b>Dirac equation</b>	$i \frac{\partial \psi}{\partial t} = (-i c \alpha \cdot \nabla + \beta \omega) \psi$	$i \frac{\partial Z}{\partial t} = (-i v A \cdot \nabla + B \omega_{1m}) Z$	<i>Complex velocity equation</i>
<b>The coefficients</b>	$\alpha$ and $\beta$	$A$ and $B$	<i>Coefficients</i>
<b>Property</b>	$\alpha_i \alpha_j + \alpha_j \alpha_i = 0$	$A_\theta \cdot A_\varphi + A_\varphi \cdot A_\theta = 0$	<i>Property</i>
<b>Property</b>	$\alpha_i \alpha_i + \alpha_i \alpha_i = 2$	$A_\theta \cdot A_\theta + A_\theta \cdot A_\theta = 2$	<i>Property</i>
<b>Property</b>	$\alpha_i^2 = \beta^2 = 1$	$A^2 = B^2 = 1$	<i>Property</i>
<b>Property</b>	$\alpha_i \beta + \beta \alpha_i = 0$	$AB + BA = 0$	<i>Property</i>
<b>Klein-Gordon equation</b>	$\frac{\partial^2 \psi}{\partial t^2} = [c^2 \nabla^2 - \omega^2] \psi$	$\frac{\partial^2 Z}{\partial t^2} = [v^2 \nabla^2 - \omega_{1m}^2] Z$	<i>Complex acceleration equation</i>

- The concept of hidden had been proposed during the history of quantum mechanics in different forms like hidden variables or hidden medium (De Broglie 1964), and the hidden geometry (Hestenes 1966). t' Hooft was against the hidden variables, but he proposed the idea of an information loss at the Planck scale. t' Hooft regards the base of the Hilbert space is related to a deep hidden level. In addition to that t' Hooft used in his underlying deterministic model a classical cogwheel model ( Planetary system) (t' Hooft 2016).

In the two circles kinematical model (Fig.1) and the partial observation, one may find a similarity with the concepts of hidden, the loss (information), the limit (Plank scale), planetary system.



- A logical question may arise: Is it possible to detect all the systems by using two light beams of different wave length? The logical answer is yes. however, is that possible experimentally for a system in microscopic scale? In this kinematical model, the photonic nature of the detecting light is not considered.
- The theoretical predictions are the results of a theoretical model, which is based on the observable world. When the observable world is different from the real (actual) world, a problem arises. As we mentioned above, the observation problem makes a barrier, which will then lead to significant errors and inconsistencies between the experimental results and the theoretical predictions. We think the problems, those described by Hossenfelder as lost in mathematics (Hossenfelder, 2018a) or as the non-normal stagnation in the present phase in the foundations of physics (Hossenfelder, 2018b), are related to the observation problem.

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