International Journal of Quantum Foundations **4** (2018) 223 - 234

Original Paper

From a classical model to an analogy of the relativistic quantum mechanics forms - II

Mohammed Sanduk

Department of Chemical and Process Engineering, University of Surrey, Guildford, GU2 7XH, UK. *E-Mail: m.sanduk@surrey.ac.uk*

Received: 1 August 2018 / Accepted: 22 September 2018 / Published online: 29 September 2018

Abstract: In the last article, an approach was developed to form an analogy of the wave function and derive analogies for both the mathematical forms of the Dirac and Klein-Gordon equations. The analogies obtained were the transformations from the classical real model forms to the forms in complex space. The analogous of the Klein-Gordon equation was derived from the analogous Dirac equation as in the case of quantum mechanics. In the present work, the forms of Dirac and Klein-Gordon equations were derived as a direct transformation from the classical model. It was found that the Dirac equation form may be related to a complex velocity equation. The Dirac's Hamiltonian and coefficients correspond to each other in these analogies. The Klein-Gordon equation form may be related to the complex acceleration equation. The complex acceleration equation can explain the generation of the flat spacetime. Although this approach is classical, it may show a possibility of unifying relativistic quantum mechanics and special relativity in a single model and throw light on the undetectable æther.

Keywords: Dirac equation, complex vector, Emergent quantum mechanics, Emergent space time, Quantum mechanics and special relativity unification, Quantum mechanics underpinning. Quantum foundation.

1. Introduction

In the last article, the model of two rolling circles (Fig.1) was used to show an analogy of relativistic quantum mechanics with the development of a partial observation technique, which was considered as a lab observation [1].

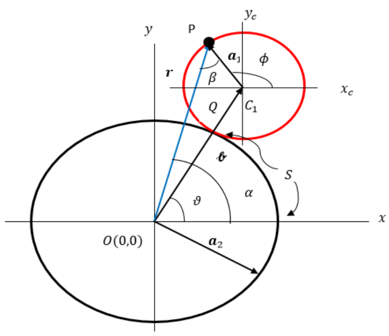

Figure 1. The real model. Rolling circles model [1].

International Journal of Quantum Foundations **4** (2018) 223 - 234

This kinematical model is used to derive an analogous kinematical form (without $\hbar$) of the Dirac's equation. The partial observation acts as a transformation technique from the real classical model to a complex analogous to quantum models. In that attempt [1], a kinematical form of the Klein-Gordon equation was derived from the analogous Dirac equation as in the case of quantum mechanics derivation and not as a direct transformation from the classical model.

This attempt [1] showed that the classical gear model with the partial observation technique might lead to the derivation of forms analogous to the Hamiltonian of relativistic quantum mechanics. In other words, this work may offer a model analogous to relativistic quantum mechanics.

In the present work, the forms of Dirac and Klein-Gordon equations will be derived as a direct transformation from the classical model (Fig.1) under the effect of partial observation. However, this work is not in quantum mechanics.

2. The system and partial observation

Considering the polar vector of point $P(\boldsymbol{r}, \alpha)$ in Fig.1, the position vector $\boldsymbol{r}$ is:

$$\boldsymbol{r} = \mathscr{b}\left\{\cos(\vartheta - \phi + \beta) \pm \sqrt{-sin^2(\vartheta - \phi + \beta) + (\frac{a_1}{\mathscr{b}})^2}\right\}. \quad (1)$$

For generality, consider $a_1 < a_2$. Owing to the rolling of the circles, the motion of a point P traces out an epicycloid-curve (hypercycloid) trajectory. The ratio of the system is:

$$\frac{a_2}{a_1} = \frac{\phi}{\vartheta} = \frac{\omega_1}{\omega_2} = \mu > 1. \quad (2)$$

Eq. (2) looks like a gear ratio. In classical physics, observable distinguishability is related to optical resolution (Rayleigh criterion). Spatial resolution (d_λ) is the minimum linear distance between two distinguishable points [2], and this is the same for angular frequency (ω_λ).

Let us say that there are two parts of different sizes in a system (like that in Fig.1). In this system, the two parts consist of different sizes (Eq.(2)). To recognize the system using monochromatic light (λ, f), d_λ is related to the wavelength, similar to how ω_λ is related to the light frequency (f). The system is fully observed (seen) when:

$$d_\lambda \ll a_1 \ll a_2, \quad (3)$$

and according to the ratio (2):

$$\omega_1 \gg \omega_2 \gg \omega_\lambda. \quad (4)$$

Within these conditions, the lab observer recognizes a classical physical system (fully determined), and all the quantities of the system are said to be physical and can be measured.

A concept of partial observation has been proposed in Ref. [15, 17]. This case may be for a system of many parts; however, some of these parts cannot be distinguished, while the others can be. Based on the resolution limit, the system (Fig.1) is partially resolved under the following conditions:

$$a_1 \ll d_\lambda \ll a_2, \tag{5a}$$

and

$$\omega_1 \gg \omega_\lambda \gg \omega_2. \tag{5b}$$

Inequalities (5a, 5b) describe the inability to resolve a_1 and ω_2 (missing data), whereas a_2 and ω_1 can be resolved. Partial resolution refers to the inability to completely resolve the kinematical system. Then the quantities measured by the lab observer are:

$$\boldsymbol{a}_1 \neq \boldsymbol{a}_{1m} = 0 \text{ , } \omega_2 \neq \omega_{2m} = 0 \text{ and } X_m = 0 \text{ ,} \tag{6}$$

and

$$\boldsymbol{b} \rightarrow \boldsymbol{a}_2 = \boldsymbol{a}_{2m}. \tag{7}$$

The subscript m indicates the resolved (measured) values. The ratio μ (Eq. (2)) is a big number ($\mu \gg 1$). When a_1 cannot be detected, then the angle β also cannot be detected. Then:

$$\omega_{\beta m} = \frac{\partial \beta}{\partial t} = 0. \tag{8}$$

Since $\alpha = \phi - \beta$, and Eq. (10d) then:

$$\omega_{\phi m} = \omega_{\alpha m} = \omega_{1m}. \tag{9}$$

where the existence of angle β is related to the recognition of the small circle of a_1. Additionally, the frequency of the angular motion can be detected indirectly (not related to rotation, which is unobservable), and the angular frequency can be measured ($\omega = 2\pi f$).

According to the observation limits mentioned above, the classical forms are subjected to the following steps:

1- Substituting the observation limits in the classical form

2- Rearranging the obtained form in terms of the complex vector

3- Presenting the obtained form in terms of mathematical operators.

This approach represents the expression of the lab observer. This technique will be applied as a transformation from the real to complex space [1].

3. The position vector

The abovementioned kinematical model (Fig.1) can be used to reformulate Eq. (1), which, in terms of circular motion, becomes:

$$\boldsymbol{r} = (\boldsymbol{a}_2 + \boldsymbol{b}\sqrt{X})\left\{\cos\left(\boldsymbol{k}_2 \cdot \boldsymbol{s} - \omega_1 t + \omega_\beta t\right) \pm \sqrt{-sin^2\left(\boldsymbol{k}_2 \cdot \boldsymbol{s} - \omega_1 t + \omega_\beta t\right) + X}\right\}. \quad (10),$$

where $\boldsymbol{s}$, ω_1, and ω_β represent the arc length made by point Q, the angular velocity of point P, and $\omega_\beta = d\beta/dt$, respectively. Eq. (10) gives a full description of the location of the point due to its movement (trajectory). The appeared ratio in Eq.(1) is:

International Journal of Quantum Foundations **4** (2018) 223 - 234

$$X \equiv \left(\frac{|\boldsymbol{a}_1|}{|\boldsymbol{b}|}\right)^2 = \left(\frac{a_1}{b}\right)^2. \tag{11}$$

The substitution of Eqs. (6, 7 and 8) into Eq. (10) yields:

$$\boldsymbol{\mathcal{Z}}(\boldsymbol{s},t) = \boldsymbol{a}_{2m}\left\{\cos\left(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t\right) \pm \sqrt{-sin^2(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)}\right\}. \tag{12}$$

The real $\boldsymbol{r}$ transforms to complex $\boldsymbol{\mathcal{Z}}$. This form of Eq. (12) can be rewritten as (in Eulerian form):

$$\boldsymbol{\mathcal{Z}}(\boldsymbol{s},t) = \boldsymbol{a}_{2m}\exp \pm i(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t). \tag{13}$$

The equations (12, 13) are for a complex vector (z-complex vector). Eq. (13) shows a combination of the real vector ($\boldsymbol{a}_{2m}$) and complex phase factor ($\exp \pm i(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)$) to form the z-complex vector.

4. The equation of velocity

From the trajectory equation (Eq.(10)), the equation of velocity is:

$$\frac{\partial\boldsymbol{r}(r,t,X)}{\partial t} = \frac{\partial(\boldsymbol{a}_2+\boldsymbol{b}\sqrt{X})}{\partial t}\left\{cos\left(\boldsymbol{k}_2\cdot\boldsymbol{s}-\omega_1 t+\omega_\beta t\right) \pm \sqrt{-sin^2\left(\boldsymbol{k}_2\cdot\boldsymbol{s}-\omega_1 t+\omega_\beta t\right)+X}\right\} + (\boldsymbol{a}_2 + \boldsymbol{b}\sqrt{X})\left\{-(-\omega_1+\omega_\beta)sin\left(\boldsymbol{k}_2\cdot\boldsymbol{s}-\omega_1 t+\omega_\beta t\right) \pm \left(\frac{1}{2}\right)\frac{(-2)(-\omega_1+\omega_\beta)\,sin(\boldsymbol{k}_2\cdot\boldsymbol{s}-\omega_1 t+\omega_\beta t)cos\left(\boldsymbol{k}_2\cdot\boldsymbol{s}-\omega_1 t+\omega_\beta t\right)+\frac{\partial X}{\partial t}}{\sqrt{-sin^2\left(\boldsymbol{k}_2\cdot\boldsymbol{s}-\omega_1 t+\omega_\beta t\right)+X}}\right\} \tag{14}$$

4.1 Unit vectors

There are two angular rotations ω_1 and ω_2, and the unit vectors of these rotations are $\hat{\boldsymbol{e}}_\phi$ and $\hat{\boldsymbol{e}}_\vartheta$, respectively. The signs $\pm$ in Eqs.(1, 10, and 14) are related to the two possibilities of rotation of the point; in other words, it is related to $\hat{\boldsymbol{e}}_\phi$. The signs show the direction of angular motion (ϕ), and the unit vector $\hat{\boldsymbol{e}}_\phi$ is the axis-angle vector.

The dot product of $\hat{\boldsymbol{e}}_\vartheta$ with any perpendicular unit vector let $\hat{\boldsymbol{e}}_\varphi$is:

$$\hat{\boldsymbol{e}}_\vartheta\cdot\hat{\boldsymbol{e}}_\varphi + \hat{\boldsymbol{e}}_\vartheta\cdot\hat{\boldsymbol{e}}_\varphi = 0. \tag{15}$$

The same is for:

$$\hat{\boldsymbol{e}}_\vartheta\cdot\hat{\boldsymbol{e}}_\vartheta + \hat{\boldsymbol{e}}_\varphi\cdot\hat{\boldsymbol{e}}_\varphi = 2. \tag{16}$$

International Journal of Quantum Foundations **4** (2018) 223 - 234

The square of the unit vectors is:

$$\hat{\boldsymbol{e}}_\vartheta \cdot \hat{\boldsymbol{e}}_\vartheta = \hat{\boldsymbol{e}}_\phi \cdot \hat{\boldsymbol{e}}_\phi = 1. \quad (17)$$

The $\hat{\boldsymbol{e}}_\phi$ and $\hat{\boldsymbol{e}}_\vartheta$ are non-commutative:

$$\hat{\boldsymbol{e}}_\vartheta \times \hat{\boldsymbol{e}}_\phi + \hat{\boldsymbol{e}}_\phi \times \hat{\boldsymbol{e}}_\vartheta = 0. \quad (18)$$

4.2 The effect of partial observation

For the lab observer who is assumed to be under the conditions of partial observation (Eqs. (6, 7 and 8)), the equation of velocity Eq. (14) transforms to (Appendix A):

$$i\frac{\partial \boldsymbol{\mathcal{Z}}}{\partial t} = (-iv\boldsymbol{A} \cdot \nabla + B\omega_{1m})\boldsymbol{\mathcal{Z}}\,, \quad (19)$$

where $\boldsymbol{A}$ and B are the coefficients related to the rotation of the system (Appendix A). The lab observer cannot recognise the rotations of the system, and, mathematically, $\boldsymbol{A}$ and B are not the normal unit vector. Thus, the properties of the rotation unit vectors mentioned above will not be considered by the observer as rotation vectors. In reality, $\boldsymbol{A}$ and B are related to the rotations of the system but are not unit vectors. As in the Eqs. (15, 16, 17 and 18), one can find that:

$$(\pm i\hat{\boldsymbol{e}}_\vartheta) \cdot (\pm i\hat{\boldsymbol{e}}_\varphi) + (\pm i\hat{\boldsymbol{e}}_\vartheta) \cdot (\pm i\hat{\boldsymbol{e}}_\varphi) = 0, \quad (20)$$

and

$$(\pm i\hat{\boldsymbol{e}}_\vartheta) \cdot (\pm i\hat{\boldsymbol{e}}_\vartheta) + (\pm i\hat{\boldsymbol{e}}_\varphi) \cdot (\pm i\hat{\boldsymbol{e}}_\varphi) = 2. \quad (21)$$

The square of B is

$$B^2 = 1, \quad (22)$$

and for $\boldsymbol{A}$

$$\boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} = \|\boldsymbol{A}\|\|\boldsymbol{A}\| = 1. \quad (23)$$

Accordingly,

$$\boldsymbol{A}^2 = B^2 = 1. \quad (24)$$

Finally (since B is not a vector, then in dealing with the $\boldsymbol{A}$ and B in the matrix form),

$$\boldsymbol{A} \times B + B \times \boldsymbol{A} = 0. \quad (25)$$

International Journal of Quantum Foundations **4** (2018) 223 - 234

5. The equation of acceleration

The second derivative of the $\mathcal{Z}$ **is:**

$$\frac{\partial^2 \boldsymbol{r}(r,t,X)}{\partial t^2} = \frac{\partial^2(\boldsymbol{a}_2 + \boldsymbol{b}\sqrt{X})}{\partial t^2}\left\{\cos\Theta \pm \sqrt{-\sin^2\Theta + X}\right\} + \frac{\partial(\boldsymbol{a}_2 + \boldsymbol{b}\sqrt{X})}{\partial t}\left\{-(-\omega_1+\omega_\beta)\sin\Theta \pm \frac{1}{2}\frac{(-2)(-\omega_1+\omega_\beta)\sin\Theta\cos\Theta + \frac{\partial X}{\partial t}}{\sqrt{-\sin^2\Theta + X}}\right\} + \frac{\partial(\boldsymbol{a}_2 + \boldsymbol{b}\sqrt{X})}{\partial t}\left\{-(-\omega_1+\omega_\beta)\sin\Theta \pm \frac{1}{2}\frac{(-2)(-\omega_1+\omega_\beta)\sin\Theta\cos\Theta + \frac{\partial X}{\partial t}}{\sqrt{-\sin^2\Theta + X}}\right\} + (\boldsymbol{a}_2 + \boldsymbol{b}\sqrt{X})\left\{-(-\omega_1+\omega_\beta)^2\cos\Theta \pm \left(\frac{1}{2}\frac{(-2)(\cos^2\Theta - \sin^2\Theta)(-\omega_1+\omega_\beta)^2 + \frac{\partial^2 X}{\partial t^2}}{\sqrt{-\sin^2\Theta + X}} + \frac{1}{2}\left(\frac{-1}{2}\right)\frac{\left[-2\sin\Theta\cos\Theta(-\omega_1+\omega_\beta) + \frac{\partial X}{\partial t}\right]\left[-2\sin\Theta\cos\Theta(-\omega_1+\omega_\beta) + \frac{\partial X}{\partial t}\right]}{\left(\sqrt[2]{-\sin^2\Theta + X}\right)^3}\right)\right\} \tag{26}$$

where

$$\Theta = \boldsymbol{k}_2 \cdot \boldsymbol{s} - \omega_1 t + \omega_\beta t$$

and

$$\frac{\partial\Theta}{\partial t} = -\omega_1 + \omega_\beta \,.$$

At the microscopic level, the lab observer deals with the complex vector $\mathcal{Z}$. Thus, we try to formulate Eq. (26) in terms of $\mathcal{Z}$ (Appendix B):

$$i^2\frac{\partial^2 \mathcal{Z}}{\partial t^2} = (-v^2\nabla^2 + {\omega_{1m}}^2)\mathcal{Z}\,. \tag{27}$$

6. Discussion

1- In comparison with the Hamiltonian of the Klein-Gordon equation, Eq. (27) shows a covariant form for the quantity in brackets.

$$H = \omega^2 = -v^2\nabla^2 + {\omega_{1m}}^2 \tag{28}$$

2- Eq. (27) is for acceleration (curved space), but it is similar to the equation used by the lab observer to describe the spinless relativistic particle (boson) in the microscopic scale and is related to the flat spacetime in macroscale.

3- The Eq. (18B)

$$i^2 \frac{\partial^2 \boldsymbol{z}}{\partial t^2} = \left\{[-v^2\nabla^2 + \omega_{1m}{}^2] \pm 2iv\omega_{1m}\hat{\boldsymbol{e}}_\theta \cdot \frac{\hat{\boldsymbol{r}}}{a_{2m}}\right\}\boldsymbol{z} \tag{18B}$$

has a third term. This term is equal to zero. This zero term is related to the angular motion of the partially observed system. The zero value maintains that this angular motion cannot be detected owing to the partial observation.

4- The third terms may be explained as a spin effect. This term is imaginary, so the spin here may be related to the concept of spin of relativistic quantum mechanics.

The spin concept plays a serious role in the loop quantum gravity theory. According to this theory, the space structure is composed of an extremely fine fabric or network woven of finite loops (spin network). The scale of these loops is of the order of a Planck length ($\lambda_P = 1.62 \times 10^{-35}$ m) [3].

5- Within the frame of Snyder's quantized spacetime [4], the Einstein Hamiltonian of a lattice of Planck length is proposed to be [5]:

$$E^2 = c^2p^2 + m^2c^4 + \alpha(\frac{c}{\hbar})^2\lambda_P{}^2p^4, \tag{29}$$

where α is a dimensionless constant. In all proposed theories, those that includes Planck length or any unit of length in the ordinary spacetime continuum arises as $\lambda \to 0$. This λ works as a boundary between the two different worlds.

In 2000 and on the base of quantum gravity, Giovanni Amelino-Camelia [6] proposed a modification of special relativity by introducing a third term to the Einstein Hamiltonian:

$$E^2 = c^2p^2 + m^2c^4 - \check{L}_p cp^2E, \tag{30}$$

where $\check{L}_p$ is related to the Planck length (λ_p).

Both the mentioned attempts used a direct combination of a term related to a microscopic realm to the macroscopic Hamiltonian terms. The justification was based on the restoration of the Hamiltonian macroscopic form when $\lambda_p \to 0$.

According to our model, there is no need for the Planck length. The third term in Eq.(18B) does not deform the normal from of the Einstein Hamiltonian.

6- For the lab observer, no "covariant æther" may exist (observed). The terminology "covariant" may be synonymous to partially observed. Therefore, there is no partially observed æther. What can be seen is the effect of the unobservable structure.

7- The system of two guided circles and one guiding circle under the partial observation can show a complex vector function of multi-dimensions [1]. If the guiding circle is attributed to the æther, then the æther may stand behind the entanglement.

International Journal of Quantum Foundations **4** (2018) 223 - 234

Appendixes

Appendix A

Under the partial observation conditions ($X = 0$ and $\omega_\beta t = 0$), Eq. (14) becomes:

$$\frac{\partial \boldsymbol{r}(r,t,0)}{\partial t} = \frac{\partial \boldsymbol{a}_{2m}}{\partial t}\left\{cos(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) \pm \sqrt{-sin^2(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)}\right\} + \boldsymbol{a}_{2m}\left\{\omega_{1m}\, sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) \pm \frac{\omega_{1m}\sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)cos(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)}{i\, sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)}\right\}. \quad (1A)$$

The vector differentiation is:

$$\frac{\partial \boldsymbol{a}_{2m}}{\partial t} = a_{2m}\omega\hat{\boldsymbol{e}}_\vartheta = v\,\hat{\boldsymbol{e}}_\vartheta\,. \quad (2A)$$

Then, Eq. (1A) becomes:

$$i\frac{\partial \boldsymbol{r}(r,t,0)}{\partial t} = (i\,v\hat{\boldsymbol{e}}_\vartheta\cdot\boldsymbol{k}_{2m})\boldsymbol{a}_{2om}\{\,cos(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) \pm\, i\,sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)\} + \omega_{1m}\boldsymbol{a}_{2m}\{i\,\,sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) \pm cos(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)\}. \quad (3A)$$

I.1 Formulation in terms of $\boldsymbol{\mathcal{Z}}$

The complex vector is:

$$\boldsymbol{\mathcal{Z}}(\boldsymbol{s},t,0) = \boldsymbol{a}_{2m}\exp \pm i(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t). \quad (13)$$

To represent Eq.(3A) in terms of $\boldsymbol{\mathcal{Z}}$, the equation can be rewritten as:

$$i\frac{\partial \boldsymbol{r}(r,t,0)}{\partial t} = (i\,v\hat{\boldsymbol{e}}_\vartheta\cdot\boldsymbol{k}_{2m})\boldsymbol{a}_{2om}\{\,cos(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) \pm\, i\,sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)\} + (\pm)\omega_{1m}\boldsymbol{a}_{2m}\{cos(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) \pm\, i\,\,sin(\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t)\}. \quad (4A)$$

In the exponential form, Eq. (4A) becomes:

$$i\frac{\partial \boldsymbol{r}(r,t,0)}{\partial t} = (i\,v\hat{\boldsymbol{e}}_\vartheta\cdot\boldsymbol{k}_{2\boldsymbol{m}})\boldsymbol{a}_{2m}\exp i \pm (\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t) + (\pm)\omega_{1m}\boldsymbol{a}_{2m}\exp i \pm (\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t), \quad (5A)$$

or

$$i\frac{\partial \boldsymbol{r}(r,t,0)}{\partial t} = [(i\,v\hat{\boldsymbol{e}}_\vartheta\cdot\boldsymbol{k}_{2\boldsymbol{m}}) + (\pm)\omega_{1m}]\boldsymbol{a}_{2m}\exp i \pm (\boldsymbol{k}_{2m}\cdot\boldsymbol{s}-\omega_{1m}t), \quad (5A)$$

or

$$i\frac{\partial \boldsymbol{r}(r,t,0)}{\partial t} = [(i\,v\hat{\boldsymbol{e}}_\vartheta\cdot\boldsymbol{k}_{2\boldsymbol{m}}) + (\pm)\omega_{1m}]\boldsymbol{\mathcal{Z}}. \quad (5A)$$

I.2 The spatial operator form

The first spatial derivative of the complex vector is:

International Journal of Quantum Foundations **4** (2018) 223 - 234

$$\nabla \boldsymbol{\mathcal{Z}} = \pm i\, \boldsymbol{k}_{2m} \boldsymbol{\mathcal{Z}}. \tag{6A}$$

The first term in Eq. (5A) can be represented in the spatial differentiation form. From Eq.(6A):

$$\boldsymbol{k}_{2m} \boldsymbol{\mathcal{Z}} \equiv \mp\, i \nabla \boldsymbol{\mathcal{Z}}.$$

Then, the first term becomes:

$$i\, v\hat{\boldsymbol{e}}_{\vartheta} \cdot \boldsymbol{k}_{2m} \boldsymbol{\mathcal{Z}} = i\, v\hat{\boldsymbol{e}}_{\vartheta} \cdot (\mp\, i \nabla \boldsymbol{\mathcal{Z}}) = -i\, v(\pm i \hat{\boldsymbol{e}}_{\vartheta}) \cdot \nabla \boldsymbol{\mathcal{Z}}. \tag{7A}$$

For the simple representation of Eq. (5A), consider

$$B = \pm 1, \text{ and } \boldsymbol{A} = \pm i\, \hat{\boldsymbol{e}}_{\vartheta}\ . \tag{8A}$$

Then Eq.(5A) becomes:

$$i \frac{\partial \boldsymbol{\mathcal{Z}}}{\partial t} = (-i\, v\boldsymbol{A} \cdot \nabla + B\omega_{1m}) \boldsymbol{\mathcal{Z}}\,. \tag{9A}$$

Appendix B

When $X = 0$,

$$\begin{aligned} \frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = {} & \frac{\partial^2 (\boldsymbol{a}_{2m})}{\partial t^2} \left\{ cos\,\Phi \pm \sqrt{-sin^2\Phi} \right\} \\ & + \frac{\partial (\boldsymbol{a}_{2m})}{\partial t} \left\{ -(-\omega_{1m})\, sin\Phi \pm \frac{-(-\omega_{1m})\, sin\Phi\, cos\Phi}{\sqrt{-sin^2\Phi}} \right\} \\ & + \frac{\partial (\boldsymbol{a}_{2m})}{\partial t} \left\{ -(-\omega_{1m}) sin\,\Phi \pm \frac{-(-\omega_{1m}) \sin\Phi cos\,\Phi}{\sqrt{-sin^2\Phi}} \right\} \\ & + (\boldsymbol{a}_{2m})(-\omega_{1m})^2 \left\{ -\cos\Phi \right. \\ & \left. \pm \left(\frac{-cos^2\Phi + sin^2\Phi}{\sqrt{-sin^2\Phi}} - \frac{[\sin\Phi cos\,\Phi][sin\,\Phi cos\,\Phi]}{\left(\sqrt[2]{-sin^2\Theta + X}\right)^3} \right) \right\} \end{aligned} \tag{1B}$$

where

$$\Phi = \boldsymbol{k}_{2m} \cdot \boldsymbol{s} - \omega_{1m} t \tag{2B}$$

Then

$$\begin{aligned} \frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = {} & \frac{\partial^2 (\boldsymbol{a}_{2m})}{\partial t^2} \left\{ cos\,\Phi \pm \sqrt{-sin^2\Phi} \right\} + 2 \frac{\partial (\boldsymbol{a}_{2m})}{\partial t} (\omega_{1m}) \left\{ sin\Phi \pm \frac{cos\Phi}{i} \right\} \\ & - (\boldsymbol{a}_{2m})(\omega_{1m})^2 \left\{ \cos\Phi \mp \left(\frac{-cos^2\Phi + sin^2\Phi}{i \sin\Phi} - \frac{[\sin\Phi cos\,\Phi]^2}{-i \sin^3\Phi} \right) \right\} \end{aligned} \tag{3B}$$

Then

International Journal of Quantum Foundations **4** (2018) 223 - 234

$$\frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = \frac{\partial^2(\boldsymbol{a}_{2m})}{\partial t^2}\{cos\,\Phi \pm i\sin\Phi\} + 2\frac{\partial(\boldsymbol{a}_{2m})}{\partial t}(\omega_{1m})\{-(\pm)i\{\,cos\Phi \pm isin\Phi\}\} - (\boldsymbol{a}_{2m})(\omega_{1m})^2\left\{\cos\Phi \mp \left(\frac{-cos^2\Phi}{i\sin\Phi} + \frac{\sin\Phi}{i} + \frac{cos^2\Phi}{i\sin\Phi}\right)\right\} \tag{4B}$$

Then

$$\frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = \frac{\partial^2(\boldsymbol{a}_{2m})}{\partial t^2}\{cos\,\Phi \pm i\sin\Phi\} + 2\frac{\partial(\boldsymbol{a}_{2m})}{\partial t}(\omega_{1m})\{-(\pm)i\{\,cos\Phi \pm isin\Phi\}\} - (\boldsymbol{a}_{2m})(\omega_{1m})^2\{\cos\Phi \pm i\ \sin\Phi\} \tag{5B}$$

Then

$$\frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = \left[\frac{\partial^2(\boldsymbol{a}_{2m})}{\partial t^2} - (\boldsymbol{a}_{2m})(\omega_{1m})^2\right]\{cos\,\Phi \pm i\sin\Phi\} + 2\frac{\partial(\boldsymbol{a}_{2m})}{\partial t}(\omega_{1m})\{-(\pm)i\{\,cos\Phi \pm isin\Phi\}\} \tag{6B}$$

The centripetal acceleration is:

$$\frac{\partial^2 \boldsymbol{a}_{2m}}{\partial t^2} = -\omega^2\boldsymbol{a}_{2m} = -\frac{v^2}{\boldsymbol{a}_{2m}} = -\frac{v^2}{a_{2m}}\hat{\boldsymbol{r}} \tag{7B}$$

The velocity

$$\frac{\partial \boldsymbol{a}_{2m}}{\partial t} = \boldsymbol{v} = v\hat{\boldsymbol{e}}_\theta \tag{8B}$$

Then

$$\frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = \left[-\frac{v^2}{a_2}\hat{\boldsymbol{r}} - (\boldsymbol{a}_{2m})(\omega_{1m})^2\right]\{cos\,\Phi \pm i\sin\Phi\} \mp 2iv\hat{\boldsymbol{e}}_\theta\omega_{1m}\{\,cos\Phi \pm isin\Phi\} \tag{9B}$$

II.1 Formulation in terms of $\boldsymbol{\mathcal{Z}}$

At the microscopic level, the lab observer deals with the complex vector $\boldsymbol{\mathcal{Z}}$. Thus, we try to formulate Eq. (9B) in terms of $\boldsymbol{\mathcal{Z}}$. The $\boldsymbol{\mathcal{Z}}$ function (Eq.13) is:

$$\boldsymbol{\mathcal{Z}} = \boldsymbol{a}_{2m}\,(cos\Phi \pm isin\Phi) \tag{10B}$$

Then divide and multiply each term on the right side of Eq.(9B) by $\boldsymbol{a}_2$:

$$\frac{\partial^2 \boldsymbol{r}(r,t,0)}{\partial t^2} = \left\{\left[-\frac{v^2}{a_{2m}}\hat{\boldsymbol{r}}\cdot\frac{1}{\boldsymbol{a}_{2m}} - (\boldsymbol{a}_{2m}\cdot\frac{1}{\boldsymbol{a}_{2m}})(\omega_{1m})^2\right] \mp 2iv\omega_{1m}\hat{\boldsymbol{e}}_\theta\cdot\frac{1}{\boldsymbol{a}_{2m}}\right\}\boldsymbol{a}_{2m}\{cos\,\Phi \pm i\sin\Phi\} \tag{11B}$$

Then Eq.(11B) becomes:

International Journal of Quantum Foundations **4** (2018) 223 - 234

$$\frac{\partial^2 \boldsymbol{Z}}{\partial t^2} = \left\{\left[-\frac{v^2}{a_{2m}}\hat{\boldsymbol{r}} \cdot \frac{1}{\boldsymbol{a}_{2m}} - (\omega_1)^2\right] \mp 2iv\omega_{1m}\hat{\boldsymbol{e}}_\theta \cdot \frac{\hat{\boldsymbol{r}}}{a_{2m}}\right\}\boldsymbol{Z}. \tag{12B}$$

Due to the orthogonality:

$$\hat{\boldsymbol{e}}_\theta \cdot \frac{1}{\boldsymbol{a}_{2m}} = \hat{\boldsymbol{e}}_\theta \cdot \frac{\hat{\boldsymbol{r}}}{a_{2m}} = 0. \tag{13B}$$

Eq. 12B becomes:

$$\frac{\partial^2 \boldsymbol{Z}}{\partial t^2} = \left(-v^2 \frac{\hat{\boldsymbol{r}}}{a_{2m}} \cdot \frac{1}{\boldsymbol{a}_{2m}} - {\omega_{1m}}^2\right)\boldsymbol{Z} = \left(v^2 \frac{1}{\boldsymbol{a}_{2m}} \cdot \frac{1}{\boldsymbol{a}_{2m}} - {\omega_{1m}}^2\right)\boldsymbol{Z}. \tag{14B}$$

II-2 The spatial operator form

Eq. (14B) can be rewritten as:

$$\frac{\partial^2 \boldsymbol{Z}}{\partial t^2} = \left(v^2 \frac{1}{\boldsymbol{a}_{2m}} \cdot \frac{-1}{\boldsymbol{a}_{2m}} - {\omega_{1m}}^2\right)\boldsymbol{Z} = \left(v^2 i^2 {\boldsymbol{k}_{2m}}^2 - {\omega_{1m}}^2\right)\boldsymbol{Z} = \left(v^2 (i\boldsymbol{k}_{2m})^2 - {\omega_{1m}}^2\right)\boldsymbol{Z}. \tag{15B}$$

From Eq. (6A)

$$\nabla \boldsymbol{Z} = (\pm) i\, \boldsymbol{k}_{2m}\boldsymbol{Z},$$

The second spatial derivative of the complex vector is:

$$\nabla^2 \boldsymbol{Z} = (i\boldsymbol{k}_{2m})^2 \boldsymbol{Z}\,. \tag{16B}$$

Then Eq. (15B) becomes:

$$\frac{\partial^2 \boldsymbol{Z}}{\partial t^2} = (v^2\nabla^2 - {\omega_{1m}}^2)\boldsymbol{Z}\,, \tag{17B}$$

or

$$i^2\frac{\partial^2 \boldsymbol{Z}}{\partial t^2} = (-v^2\nabla^2 + {\omega_{1m}}^2)\boldsymbol{Z}\,. \tag{17B}$$

Eq.(12B) becomes

$$i^2\frac{\partial^2 \boldsymbol{Z}}{\partial t^2} = \left\{[-v^2\nabla^2 + {\omega_{1m}}^2] \pm 2iv\omega_{1m}\hat{\boldsymbol{e}}_\theta \cdot \frac{\hat{\boldsymbol{r}}}{a_{2m}}\right\}\boldsymbol{Z} \tag{18B}$$

References

1. Sanduk, M. (2018). An analogy for the relativistic quantum mechanics via a model of de Broglie wave-covariant æther, *International Journal of Quantum Foundations* 4: 173–198.
2. Khare, P. and Swarup, A. (2009). Engineering Physics: Fundamentals & Modern

International Journal of Quantum Foundations **4** (2018) 223 - 234

Applications, Sudbury: Jones & Bartlett Publishers.

3. Carlo Rovelli. (2014). Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory (Cambridge Monographs on Mathematical Physics), Cambridge University Press.
4. Snyder, H.S. (1947). *Phys. Rev*. 71: 38–41.
5. Glinka, L.A. (2010). *Apeiron* 17(4): 223–242.
6. Amelino-Camelia, G. (2000). Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale, arXiv: gr-qc/0012051v2 26.