Original Paper

Duration and its Relationship to the Structure of Space-Time II

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Abstract: We assign to the radiation vacuum the role of a universal observer with a corresponding universal clock. By demanding that the thermal clock of a gravitationally accelerated observer in its local rest frame marches in step with the universal one, we derive relations between energy content and geometry of space-time.

Keywords: Quantum Physics, Relativity, Entropy, Minkowski space-time, Lorentz space-time

1. Introduction

In [1] it is shown how a measurement on a quantum-system ψ with (von Neumann) entropy S_ψ triggers the environment, which is assumed to be in equilibrium at temperature T, to "tick" like a clock with a frequency τ given by

$$\tau = \frac{4}{h} S_\psi \langle E_\nu \rangle. \tag{1}$$

$\langle E_\nu \rangle$ denotes the average energy of a radiation mode (harmonic oscillator) with $h\nu \ll k_B T$ and hence $k_B T \approx \langle E_\nu \rangle$. The proportionality factor $k_B$is the Boltzmann-constant and h denotes, as usual, Planck's constant. The frequency τ represents the inverse of the duration, which it takes the environment to move into an orthogonal state with available process-energy of $\bar{E} = S_\psi \langle E_\nu \rangle$. We can think of this process to represent a thermal clock with period $\Delta = \frac{1}{\tau}$. The environment acts as a universal observer in its own right, which notices that a measurement has a result with certainty. In [1] it is further shown that for the universal observer the geometry of space-time turns out to be consistent with the Minkowski structure. The analysis in [1] is done in non-accelerated frames and it seems natural to extend the ideas to accelerated frames, which is the purpose of this paper.

2. Accelerated frames

We consider a uniformly accelerated observer in Minkowski space-time, who moves with acceleration κ in x-direction. We chose a co-moving coordinate system, which is defined in the wedge, limited by $|x| = t$, and given by the transformations

$$x = \varrho \cosh(\kappa\vartheta)\,, t = \varrho \sinh(\kappa\vartheta)\,, \quad \varrho \geq 0, -\infty < \vartheta < \infty. \tag{2}$$

The line-element is

$$ds^2 = \left(\frac{\kappa\varrho}{c^2}\right)^2 c^2 d\vartheta^2 - d\rho^2 - dy^2 - dz^2. \tag{3}$$

These are so-called Rindler-coordinates [2]. At $t = 0$, the observers are at rest in their respective local inertial frames and equation (1) takes the form

$$d\tau = \frac{4}{h} S_\psi \langle E_\nu \rangle \frac{\kappa\varrho}{c} d\vartheta. \tag{4}$$

In the sequel we let ψ be a system with $S_\psi = 1$. A thermal clock in the local inertial frame at $\varrho_1 = \frac{c^2}{\kappa}$ and another one stationary at $\varrho_2 = 1$, say, are synchronized, if

$$d\tau_1 = d\tau_2.^1 \tag{5}$$

By using $\langle E_\nu \rangle \approx k_B T$ we get

$$d\tau_1 = \frac{4}{h} k_B T_{\left(\varrho = \frac{c^2}{\kappa}\right)} \cdot c \cdot d\vartheta = d\tau_2 = \frac{4}{h} k_B T_{(\rho=1)} \cdot \frac{\kappa}{c} \cdot d\vartheta.$$

With $T_\kappa := T_{\left(\varrho = \frac{c^2}{\kappa}\right)}$ we arrive at

$$\nu_\kappa = \frac{4}{h} k_B T_\kappa \cdot \frac{c^2}{\kappa} = \frac{4}{h} k_B T_{(\varrho=1)}. \tag{6}$$

This is a special case of the Tolman-Ehrenfest effect [3,4]. The equilibrium condition (6) forces different temperatures upon different stationary observers in accelerated coordinates. Conditions (5) and (6) are relational in nature and there is a "gauge" freedom by choosing $T_{(\varrho=1)}$. So far we work in

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[1] For synchronization it is sufficient to demand $d\tau 1 = \delta d\tau 2$ for some $\delta \in \mathbb{R}+$. We set $\delta=1$.

one chart. We can generalize (5) and (6) by demanding them to hold for different charts $\kappa_1 \neq \kappa_2$ and hence generalize it beyond flat Minkowski space-time.

For this purpose we consider a specific observer, namely a harmonic oscillator with frequency ν in vacuum. For its average energy we have in the local rest frame

$$\langle E_\nu \rangle = \frac{1}{2} h\nu. \tag{7}$$

With $\omega = 2\pi\nu$ and λ being the wave-length, the acceleration is $\kappa = \lambda\omega^2$ and the left hand side of (6) turns into

$$v_\omega = \frac{4}{h} \cdot \frac{1}{2} \hbar\omega \cdot \frac{c^2}{\lambda\omega^2} = \frac{c^2}{\lambda\pi\omega} = \frac{c^2}{2c\pi^2} = \frac{c}{2\pi^2}. \tag{8}$$

v_ω is independent of ω. If the environment runs through a large series of positions instead of flipping back and forth, then the corresponding maximal frequency $\tilde{\tau}$ turns out to be $\tilde{\tau} = \frac{\tau}{2}$ [5]. Therefore we can define the equilibrium (5) between a monotonous clock in the local rest frame of an observer with acceleration κ and the harmonic oscillator, locally at rest in vacuum, by setting $\frac{1}{2} v_\kappa = v_\omega$. This implies

$$\frac{4}{h} k_B T_\kappa \cdot \frac{c}{\kappa} = \frac{1}{\pi^2}. \tag{9}$$

From (9) we get for the corresponding temperature T_κ

$$T_\kappa = \frac{\hbar\kappa}{2\pi c k_B}. \tag{10}$$

Expression (10) is the Unruh-Davies [6,7] temperature.

We have seen that, if we take an electromagnetic mode, locally at rest in vacuum, as a universal reference clock and demand that the thermal clock of any other accelerated observer in its local rest frame ticks in step[2], then the accelerated observer necessarily experiences a temperature $T_\kappa > 0$, which is independent of the frequency ν of the mode. In fact the universal clock is just light, which of course moves at the constant speed c regardless of its frequency ν (8). We will now investigate what we find, if we additionally make a special choice for κ.

3. Gravity

[2] The thermal clock is supposed to be monotonously running, while the harmonic oscillator is periodically flipping between two states.

Let a mass M reside at a point p in an almost flat space-time [3]. The line-element in the static Newtonian limit, where no part is moving, turns out to be [8]

$$ds^2 = \left(1 + \frac{2\varphi}{c^2}\right)^2 (cdt)^2 - dx^2 - dy^2 - dz^2, \tag{11}$$

where $\varphi(r) = -\frac{GM}{r}$ is the Newtonian potential and G denotes the gravitational constant. At any fixed radius $r = R, R \neq 0$, the metric components are finite and the surface gravity g_R is the ordinary Newtonian acceleration

$$g_R := -\partial_r g_{00|R} = -\frac{2}{c^2}\partial_r \varphi_{|R} = \frac{2GM}{c^2 R^2}. \tag{12}$$

We now recall equation (9), which takes with $\kappa = g_R$ the form

$$\frac{4}{h} k_B T_{g_R} \frac{c}{g_R} = \frac{1}{\pi^2}. \tag{13}$$

Plugging (12) into (13), with $A_R = 4\pi R^2$ denoting the surface area of the sphere S_R of radius R, results in

$$\frac{4c^2}{h} k_B T_{g_R} \frac{cR^2}{2GM} = \frac{2c^2}{h} k_B T_{g_R} \frac{A_R}{4\pi} \frac{c}{GM} = \frac{1}{\pi^2}. \tag{14}$$

By defining the Planck length $l_P = \sqrt{\frac{G\hbar}{c^3}}$ and by using the relativistic relation for total energy in the rest frame $E = Mc^2$, (14) turns after a short calculation into

$$k_B T_{g_R} \frac{A_R}{4 l_P^2} = E. \tag{15}$$

Equation (15) has the form of an energy-entropy relation $TS = Q$, which allows the formal identification

$$S = \frac{k_B A_R}{4 l_P^2}. \tag{16}$$

To proceed further, we plug the corresponding expression for temperature (10) $T_{g_R} = \frac{\hbar g_R}{2\pi c k_B}$ into (15) to arrive at

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[3] So the mass M is not too big.

$$g_R A_R = \frac{8\pi l_P^2}{c\hbar} E = \frac{8\pi G}{c^4} E. \tag{17}$$

If we introduce the Gaussian curvature K we can write (17) with some suitable constant $\alpha \in \mathbb{R}$ in the more suggestive form

$$\alpha \int_{S_R} K dA = \frac{8\pi G}{c^4} E.$$

If $\mathcal{S}$ denotes a topological sphere with $S_R \subset \mathcal{S}$ ($\mathcal{S} \subset S_R$) and if there is no additional energy within $\mathcal{S}$ (S_R), then there holds [4]

$$\alpha \int_{\mathcal{S}} K dA = \frac{8\pi G}{c^4} E. \tag{18}$$

Equation (18) is a relationship between geometry, mass and energy in 3-space.

Equation (17) was found for the special case of the Newtonian approximation (11). It can directly be extended to general static Lorentz four-manifolds.

We consider an asymptotically flat, static background with global time-like Killing vector field ξ^a. The generalization of Newton's potential φ can be defined by

$$\varphi = \frac{1}{2} log(-\xi^a \xi_a). \tag{19}$$

The exponential e^φ is the red-shift factor, which defines a foliation of space-time in space-like surfaces $\mathcal{S}$ of constant red shift. In this set-up a free falling particle will have a four-acceleration perpendicular to $\mathcal{S}$ given by [9]

$$a^b = -\nabla^b \varphi. \tag{20}$$

Recalling (12), the left hand side of (17) formally turns into the more general expression[5]

$$g_R A_R \to \int_{\mathcal{S}} e^\varphi \nabla\varphi \cdot dA. \tag{21}$$

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[4] For example as a consequence of the Gauss-Bonnet theorem.

[5] The factor e^φ is used because the space-time is asymptotically flat and quantities, including temperature, are measured relative to $\varphi = 0$.

(21) is (modulo constants) exactly the expression for the Komar mass M [9] and we indeed get, by accounting for the constants, the equivalent equation to (17)

$$\frac{1}{2}\int_{\mathcal{S}} e^{\varphi}\nabla\varphi \cdot dA = \frac{4\pi G}{c^2}\cdot M = \frac{4\pi G}{c^4}E. \tag{22}$$

For completeness sake we also sketch the steps as indicated in [10] to the derivation of the analogue to (18), namely Einstein's equations from (22).
Re-expressed in terms of the Killing vector ξ^a there holds [9]

$$\int_{\mathcal{S}} dx^a \wedge dx^b \epsilon_{abcd}\nabla^c\xi^d = \frac{8\pi G}{c^4}E\,. \tag{23}$$

By Stokes theorem and the identity $\nabla^a\nabla_a\xi^b = -R_a^b\xi^a$ (23) turns into

$$\int_{\Sigma} R_{ab}n^a\xi^b dV = \frac{8\pi G}{c^4}E, \tag{24}$$

where Σ is a volume bounded by $\mathcal{S}$. Since the Ricci tensor R_{ab} equals zero in a massless region, relation (24) holds for any boundary surface $\mathcal{S}$ of Σ, as long as Σ comprises all the matter. By writing the right hand side as an appropriate integral over components[6] of the energy-stress tensor T_{ab} we get

$$\int_{\Sigma} R_{ab}n^a\xi^b dV = \frac{8\pi G}{c^4}\int_{\Sigma}\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)n^a\xi^b dV. \tag{25}$$

Finally, by considering a small, almost flat space-time region and imposing that, if matter m crosses the screen, then the Komar integral changes by that amount, (23) can be shown to hold for all (approximate) Killing vectors ξ^a and screens $\mathcal{S}$ with normal vector n^a. Therefore

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \frac{1}{2}Tg_{ab}\right). \tag{26}$$

A similar approach was taken in [11] by using null-screens.

4. Conclusion

[6] The energy field should be source free, for instance.

As already shown in [1] for flat space-time, we have seen in the present work that there is a close relationship between the general geometry of space-time and the idea to use light as a universal clock. The accelerated case thereby clearly bases on the results in flat space-time and the corresponding special relativistic consequences. We imagine that the environment is filled with electromagnetic radiation and show how this environment can serve as a universal observer with a corresponding universal clock. By synchronizing other thermal clocks, in particular those freely falling in a gravitational field, with the universal clock, we are directly led to relationships (17), (18) and ultimately to the covariant extensions (22) and (24) between the geometry of space-time and its energy content. This is additional evidence to the one found in [1] that the kinematics of material bodies and hence the geometry of space and time are closely linked to observing the material world in a specific way by light and using its implied natural clock as a reference clock. Intuitively this has probably always been clear, because human beings are physiologically conditioned to experience the material world to a large degree by the help of light.

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