

Duration and its Relationship to the Structure of Space-Time II

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Abstract

We assign to the radiation vacuum the role of a universal observer with a corresponding universal clock. By demanding that the thermal clock of a gravitationally accelerated observer in its local rest frame marches in step with the universal one, we derive relations between energy content and geometry of space-time.

Key Words: Quantum Physics, Relativity, Entropy, Minkowski space-time, Lorentz space-time

1. Introduction

In [1] it is shown how a measurement on a quantum-system ψ with (von Neumann) entropy S_ψ triggers the environment, which is assumed to be in equilibrium at temperature T , to “tick” like a clock with a frequency τ given by

$$\tau = \frac{4}{h} S_\psi \langle E_\nu \rangle. \quad (1)$$

$\langle E_\nu \rangle$ denotes the average energy of a radiation mode (harmonic oscillator) with $h\nu \ll k_B T$ and hence $k_B T \approx \langle E_\nu \rangle$. The proportionality factor k_B is the Boltzmann-constant and h denotes, as usual, Planck’s constant. The frequency τ represents the inverse of the duration, which it takes the environment to move into an orthogonal state with available process-energy of $\bar{E} = S_\psi \langle E_\nu \rangle$. We can think of this process to represent a thermal clock with period $\Delta = \frac{1}{\tau}$. The environment acts as a universal observer in its own right, which notices that a measurement has a result with certainty. In [1] it is further shown that for the universal observer the geometry of space-time turns out to be consistent with the Minkowski structure. The analysis in [1] is done in non-accelerated frames and it seems natural to extend the ideas to accelerated frames, which is the purpose of this paper.

2. Accelerated frames

We consider a uniformly accelerated observer in Minkowski space-time, who moves with acceleration κ in x -direction. We chose a co-moving coordinate system, which is defined in the wedge, limited by $|x| = t$, and given by the transformations

$$x = \varrho \cosh(\kappa\vartheta), t = \varrho \sinh(\kappa\vartheta), \quad \varrho \geq 0, -\infty < \vartheta < \infty. \quad (2)$$

The line-element is

$$ds^2 = \left(\frac{\kappa\varrho}{c^2}\right)^2 c^2 d\vartheta^2 - d\rho^2 - dy^2 - dz^2. \quad (3)$$

These are so-called Rindler-coordinates [2]. At $t = 0$, the observers are at rest in their respective local inertial frames and equation (1) takes the form

$$d\tau = \frac{4}{h} S_\psi \langle E_\nu \rangle \frac{\kappa \varrho}{c} d\vartheta. \quad (4)$$

In the sequel we let ψ be a system with $S_\psi = 1$. A thermal clock in the local inertial frame at $\varrho_1 = \frac{c^2}{\kappa}$ and another one stationary at $\varrho_2 = 1$, say, are synchronized, if

$$d\tau_1 = d\tau_2. \quad (5)$$

By using $\langle E_\nu \rangle \approx k_B T$ we get

$$d\tau_1 = \frac{4}{h} k_B T_{\left(\varrho = \frac{c^2}{\kappa}\right)} \cdot c \cdot d\vartheta = d\tau_2 = \frac{4}{h} k_B T_{(\varrho=1)} \cdot \frac{\kappa}{c} \cdot d\vartheta.$$

With $T_\kappa := T_{\left(\varrho = \frac{c^2}{\kappa}\right)}$ we arrive at

$$v_\kappa = \frac{4}{h} k_B T_\kappa \cdot \frac{c^2}{\kappa} = \frac{4}{h} k_B T_{(\varrho=1)}. \quad (6)$$

This is a special case of the Tolmann-Ehrenfest effect [3,4]. The equilibrium condition (6) forces different temperatures upon different stationary observers in accelerated coordinates. Conditions (5) and (6) are relational in nature and there is a “gauge” freedom by choosing $T_{(\varrho=1)}$. So far we work in one chart. We can generalize (5) and (6) by demanding them to hold for different charts $\kappa_1 \neq \kappa_2$ and hence generalize it beyond flat Minkowski space-time.

For this purpose we consider a specific observer, namely a harmonic oscillator with frequency ν in vacuum. For its average energy we have in the local rest frame

$$\langle E_\nu \rangle = \frac{1}{2} h\nu. \quad (7)$$

With $\omega = 2\pi\nu$ and λ being the wave-length, the acceleration is $\kappa = \lambda\omega^2$ and the left hand side of (6) turns into

$$v_\omega = \frac{4}{h} \cdot \frac{1}{2} \hbar\omega \cdot \frac{c^2}{\lambda\omega^2} = \frac{c^2}{\lambda\pi\omega} = \frac{c^2}{2c\pi^2} = \frac{c}{2\pi^2}. \quad (8)$$

v_ω is independent of ω . If the environment runs through a large series of positions instead of flipping back and forth, then the corresponding maximal frequency $\tilde{\tau}$ turns out to be $\tilde{\tau} = \frac{\tau}{2}$ [5]. Therefore we can define the equilibrium (5) between a

¹ For synchronization it is sufficient to demand $d\tau_1 = \delta d\tau_2$ for some $\delta \in \mathbb{R}^+$. We set $\delta = 1$.

monotonous clock in the local rest frame of an observer with acceleration κ and the harmonic oscillator, locally at rest in vacuum, by setting $\frac{1}{2}v_\kappa = v_\omega$. This implies

$$\frac{4}{\hbar}k_B T_\kappa \cdot \frac{c}{\kappa} = \frac{1}{\pi^2}. \quad (9)$$

From (9) we get for the corresponding temperature T_κ

$$T_\kappa = \frac{\hbar\kappa}{2\pi c k_B}. \quad (10)$$

Expression (10) is the Unruh-Davies [6,7] temperature.

We have seen that, if we take an electromagnetic mode, locally at rest in vacuum, as a universal reference clock and demand that the thermal clock of any other accelerated observer in its local rest frame ticks in step², then the accelerated observer necessarily experiences a temperature $T_\kappa > 0$, which is independent of the frequency ν of the mode. In fact the universal clock is just light, which of course moves at the constant speed c regardless of its frequency ν (8). We will now investigate what we find, if we additionally make a special choice for κ .

3. Gravity

Let a mass M reside at a point p in an almost flat space-time³. The line-element in the static Newtonian limit, where no part is moving, turns out to be [8]

$$ds^2 = \left(1 + \frac{2\varphi}{c^2}\right)^2 (cdt)^2 - dx^2 - dy^2 - dz^2, \quad (11)$$

where $\varphi(r) = -\frac{GM}{r}$ is the Newtonian potential and G denotes the gravitational constant. At any fixed radius $r = R, R \neq 0$, the metric components are finite and the surface gravity g_R is the ordinary Newtonian acceleration

$$g_R := -\partial_r g_{00}|_R = -\frac{2}{c^2} \partial_r \varphi|_R = \frac{2GM}{c^2 R^2}. \quad (12)$$

We now recall equation (9), which takes with $\kappa = g_R$ the form

$$\frac{4}{\hbar}k_B T_{g_R} \frac{c}{g_R} = \frac{1}{\pi^2}. \quad (13)$$

Plugging (12) into (13), with $A_R = 4\pi R^2$ denoting the surface area of the sphere S_R of radius R , results in

² The thermal clock is supposed to be monotonously running, while the harmonic oscillator is periodically flipping between two states.

³ So the mass M is not too big.

$$\frac{4c^2}{h} k_B T_{gR} \frac{cR^2}{2GM} = \frac{2c^2}{h} k_B T_{gR} \frac{A_R}{4\pi GM} = \frac{1}{\pi^2}. \quad (14)$$

By defining the Planck length $l_P = \sqrt{\frac{G\hbar}{c^3}}$ and by using the relativistic relation for total energy in the rest frame $E = Mc^2$, (14) turns after a short calculation into

$$k_B T_{gR} \frac{A_R}{4l_P^2} = E. \quad (15)$$

Equation (15) has the form of an energy-entropy relation $TS = Q$, which allows the formal identification

$$S = \frac{k_B A_R}{4l_P^2}. \quad (16)$$

To proceed further, we plug the corresponding expression for temperature (10) $T_{gR} = \frac{\hbar g_R}{2\pi c k_B}$ into (15) to arrive at

$$g_R A_R = \frac{8\pi l_P^2}{c\hbar} E = \frac{8\pi G}{c^4} E. \quad (17)$$

If we introduce the Gaussian curvature K we can write (17) with some suitable constant $\alpha \in \mathbb{R}$ in the more suggestive form

$$\alpha \int_{S_R} K dA = \frac{8\pi G}{c^4} E.$$

If \mathcal{S} denotes a topological sphere with $S_R \subset \mathcal{S}$ ($\mathcal{S} \subset S_R$) and if there is no additional energy within \mathcal{S} (S_R), then there holds ⁴

$$\alpha \int_{\mathcal{S}} K dA = \frac{8\pi G}{c^4} E. \quad (18)$$

Equation (18) is a relationship between geometry, mass and energy in 3-space.

Equation (17) was found for the special case of the Newtonian approximation (11). It can directly be extended to general static Lorentz four-manifolds.

We consider an asymptotically flat, static background with global time-like Killing vector field ξ^a . The generalization of Newton's potential φ can be defined by

$$\varphi = \frac{1}{2} \log(-\xi^a \xi_a). \quad (19)$$

⁴ For example as a consequence of the Gauss-Bonnet theorem.

The exponential e^φ is the red-shift factor, which defines a foliation of space-time in space-like surfaces \mathcal{S} of constant red shift. In this set-up a free falling particle will have a four-acceleration perpendicular to \mathcal{S} given by [9]

$$a^b = -\nabla^b \varphi. \quad (20)$$

Recalling (12), the left hand side of (17) formally turns into the more general expression⁵

$$g_R A_R \rightarrow \int_{\mathcal{S}} e^\varphi \nabla \varphi \cdot dA. \quad (21)$$

(21) is (modulo constants) exactly the expression for the Komar mass M [9] and we indeed get, by accounting for the constants, the equivalent equation to (17)

$$\frac{1}{2} \int_{\mathcal{S}} e^\varphi \nabla \varphi \cdot dA = \frac{4\pi G}{c^2} \cdot M = \frac{4\pi G}{c^4} E. \quad (22)$$

For completeness sake we also sketch the steps as indicated in [10] to the derivation of the analogue to (18), namely Einstein's equations from (22). Re-expressed in terms of the Killing vector ξ^a there holds [9]

$$\int_{\mathcal{S}} dx^a \wedge dx^b \epsilon_{abcd} \nabla^c \xi^d = \frac{8\pi G}{c^4} E. \quad (23)$$

By Stokes theorem and the identity $\nabla^a \nabla_a \xi^b = -R_a^b \xi^a$ (23) turns into

$$\int_{\Sigma} R_{ab} n^a \xi^b dV = \frac{8\pi G}{c^4} E, \quad (24)$$

where Σ is a volume bounded by \mathcal{S} . Since the Ricci tensor R_{ab} equals zero in a massless region, relation (24) holds for any boundary surface \mathcal{S} of Σ , as long as Σ comprises all the matter. By writing the right hand side as an appropriate integral over components⁶ of the energy-stress tensor T_{ab} we get

$$\int_{\Sigma} R_{ab} n^a \xi^b dV = \frac{8\pi G}{c^4} \int_{\Sigma} \left(T_{ab} - \frac{1}{2} T g_{ab} \right) n^a \xi^b dV. \quad (25)$$

Finally, by considering a small, almost flat space-time region and imposing that, if matter m crosses the screen, then the Komar integral changes by that amount, (23)

⁵ The factor e^φ is used because the space-time is asymptotically flat and quantities, including temperature, are measured relative to $\varphi = 0$.

⁶ The energy field should be source free, for instance.

can be shown to hold for all (approximate) Killing vectors ξ^a and screens \mathcal{S} with normal vector n^a . Therefore

$$R_{ab} = \frac{8\pi G}{c^4} \left(T_{ab} - \frac{1}{2} T g_{ab} \right). \quad (26)$$

A similar approach was taken in [11] by using null-screens.

4. Conclusion

As already shown in [1] for flat space-time, we have seen in the present work that there is a close relationship between the general geometry of space-time and the idea to use light as a universal clock. The accelerated case thereby clearly bases on the results in flat space-time and the corresponding special relativistic consequences. We imagine that the environment is filled with electromagnetic radiation and show how this environment can serve as a universal observer with a corresponding universal clock. By synchronizing other thermal clocks, in particular those freely falling in a gravitational field, with the universal clock, we are directly led to relationships (17), (18) and ultimately to the covariant extensions (22) and (24) between the geometry of space-time and its energy content. This is additional evidence to the one found in [1] that the kinematics of material bodies and hence the geometry of space and time are closely linked to observing the material world in a specific way by light and using its implied natural clock as a reference clock. Intuitively this has probably always been clear, because human beings are physiologically conditioned to experience the material world to a large degree by the help of light.

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