Bell’s theorem without inequalities: on the inception and scope of the GHZ theorem

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1. Introduction

Since its inception, fifty years ago, Bell’s theorem has had a long history not only of experimental tests but also of theoretical developments. Studying pairs of correlated quantum-mechanical particles separated in space, in a composite “entangled” state, Bell (1964) showed that the joint ascription of hidden-variables and locality to the system led to an inequality that is violated by the predictions of quantum mechanics. Fifteen years later, experiments confirmed the predictions of quantum mechanics, ruling out a large class of local realist theories.

One of the most meaningful theoretical developments that followed Bell’s work was the Greenberger-Horne-Zeilinger (GHZ) theorem, also known as Bell’s theorem without inequalities. In 1989, the American physicists Daniel Greenberger and Michael Horne, who had been working on Bell’s theorem since the late 1960s, together with Austrian physicist Anton Zeilinger, introduced a novelty to the testing of entanglement, extending Bell’s theorem in a different and interesting direction. According to Franck Laloë (2012, p. 100),

> For many years, everyone thought that Bell had basically exhausted the subject by considering all really interesting situations, and that two-spin systems provided the most spectacular quantum violations of local realism. It therefore came as a surprise to many when in 1989 Greenberger, Horne, and Zeilinger (GHZ) showed that systems containing more than two correlated particles may actually exhibit even more dramatic violations of local realism.

The trio analyzed the Einstein, Podolsky, and Rosen 1935 argument once again and were able to write what are now called Greenberger-Horne-Zeilinger (GHZ) entangled states, involving three or four correlated spin-½ particles, leading to conflicts between local realist theories and quantum mechanics. However, unlike Bell's theorem, the conflict now was not of a statistical nature, as was the case with Bell's theorem, insofar as measurements on a single GHZ state could lead to conflicting predictions with local realistic models (Greenberger, Horne & Zeilinger, 1989; Greenberger et al. 1990; Greenberger, 2002).[1] In this paper we present the history of the creation of this theorem and analyze its scope.

2. The men behind the GHZ theorem

In the early 1980s, Greeenberger, Horne, and Zeilinger began to collaborate around the subject of neutron interferometry at the Massachusetts Institute of Technology (MIT), but they came from different backgrounds concerning the research on the foundations of quantum mechanics. As early as 1969, while doing his PhD dissertation at Boston University under the supervision of Abner Shimony, Horne began to work on foundations. His challenge was to carry Bell's theorem to a laboratory test, in which he succeeded, but not alone. What is now called the CHSH paper (Clauser et al., 1969), which is the adaptation of Bell's original theorem to a real life experiment, resulted from the collaboration among Shimony and Horne with John Clauser, an experimental physicist who was independently working on the same subject at the University of California at Berkeley, and Richard Holt, who was doing his PhD at Harvard under Francis Pipkin on the same issue. The CHSH paper triggered a string of experimental tests, which continue to date, leading ultimately to the wide acknowledgment of entanglement as a new physical effect. In the early 1970s, however, Horne thought this subject was dead and turned to work with neutrons while teaching at Stonehill College, near Boston.

[1] The first paper is the original presentation of the GHZ theorem. The second paper, co-authored with Abner Shimony, contains a more detailed presentation of the theorem, its proof, and suggests possible experiments, including momentum and energy correlations among three and more photons produced through parametric down conversion. The third paper presents Greenberger's recollections of the background of the original paper.

Zeilinger's interests in foundational issues flourished while studying at the University of Vienna, and were favored by the flexible curriculum at this university at that time and by the intellectual climate of physics in Vienna – with its mix of science and philosophy –, a legacy coming from the late 19th century. In addition, working under the supervision of Helmut Rauch on neutron interferometry, he benefitted from Rauch's support to research on foundations of quantum mechanics.[2] In 1976, as a consequence of this interest shared with Rauch, Zeilinger went to a conference in Erice, Italy, organized by John Bell, fully dedicated to the foundations of quantum mechanics. Over there, while the hottest topic was Bell's experiments, Zeilinger talked about neutron interferometry. He presented a report on experiments held by Rauch's team confirming a counter-intuitive quantum prediction. This prediction states that a neutron quantum state changes its signal after a 2π rotation, only recovering the original signal after a 4π rotation (Rauch et al., 1975). Zeilinger went to Erice unaware of entanglement but came back fascinated by the subject.

Greenberger was a high-energy theorist working at the City College of New York when he decided to move to a domain where connections between quantum mechanics and gravity could be revealed. He chose neutron experiments precisely for looking for these connections and met Zeilinger and Horne at a neutron conference in Grenoble (Greenberger, 2002). In the early 1970s, Clifford Shull's laboratory at MIT became the meeting point for the GHZ trio. Shull worked on neutron scattering, which earned him the 1994 Nobel Prize. Zeilinger was there as a postdoc student and later as a Visiting Professor. Horne taught at Stonehill College and conducted research at MIT. Greenberger was always around the lab. Basic experiments with neutron interferometry were the bread and butter of the trio.

For Zeilinger, the GHZ theorem was a reward for a risky professional change. In the mid-1980s he had decided to leave neutron interferometry to build a research program in quantum and atomic optics from scratch. Zeilinger was supported by the Austrian Science Foundation and looked for the basics in the new field. Sometimes he did this through interaction with other teams, such as Leonard Mandel's in Rochester. Reasons for this choice were related to his understanding that these fields offered

[2]Anton Zeilinger, interviewed by Olival Freire, 30 June 2014, American Institute of Physics (AIP). For Rauch's research on neutron interferometry and its relation to foundational issues, see the review Rauch (2012).

more opportunities than neutron interferometry. He was increasingly attracted by the foundations of quantum physics and particularly by the features of entanglement.

The collaboration among the trio started with joint work involving only Zeilinger and Horne, and concerned quantum two-particle interference. Indeed, this topic, nowadays a hallmark of quantum light behavior, was independently exploited by two teams and arose in physics through two different and independent paths. On the one hand, via quantum optics, and on the other, via scientists who were working on neutron interferometry, such as Anton Zeilinger and Michael Horne. In 1985, unaware of the achievements in the quantum optics community, Zeilinger and Horne tried to combine the interferometry experiments they were doing with Bell's theorem. Then they suggested a new experiment with Bell's theorem using light, but instead of using correlation among polarizations (internal variables) they used linear momenta. They concluded that the quantum description of two-particle interferometry was completely analogous to the description of singlet spin state used by Bell (Horne & Zeilinger, 1985). However, they did not know how to produce such states in laboratories, because they "didn't know where to get a source that would emit pairs of particles in opposite directions." When Horne read the Ghosh-Mandel paper (Ghosh & Mandel, 1987), they sent their paper to Mandel, who reacted saying: "This is so much simpler than the way we describe it, you should publish it." They called Horne's former supervisor, Abner Shimony, and wrote the "two-particle interferometry" paper explaining "the fundamental ideas of the recently opened field of two-particle interferometry, which employs spatially separated, quantum mechanically entangled two-particle states" (Horne, Shimony & Zeilinger, 1989).[3]

While neutron interferometry had been the common ground of collaboration among Zeilinger, Greenberger, and Horne, the GHZ theorem was a result that had implications far beyond their original interest. The first intuition related to the GHZ theorem came from Greenberger, who asked Zeilinger and Horne, "Do you think there would be something interesting with three particles that are entangled? Would there be any difference, something new to learn with a three-particle entanglement?" The work matured while he spent a sabbatical in Vienna working with Zeilinger. "I

[3] The 1985 Horne and Zeilinger paper was prepared for a conference in Joensuu, Finland, dedicated to the 50th anniversary of the EPR paper. This was the first of Zeilinger's paper to deal with Bell's theorem. Interview with Michael Horne, by Joan Bromberg, 12 Sep 2002, AIP. Interview with Anton Zeilinger, by Olival Freire Jr., 30 June 2014, AIP.

have a Bell's theorem without inequalities," was the manner in which he reported his results to Horne.[4]

3. Reception of the GHZ theorem

The GHZ theorem was ready in 1986, but the result remained unpublished for three years. In 1988 Greenberger presented the proof at a conference at George Mason University (which led to the publication the following year), and in 1989 it was presented at a conference in Sicily, in honor of Werner Heisenberg, attended by N. David Mermin, from the United States, and Michael Redhead, from England, who began to publicize it (Mermin, 1990; Clifton et al., 1991). Zeilinger recalls when he met Bell at a conference in Amherst, in 1990, how enthusiastic Bell was about the result. Unfortunately, Bell's untimely death in that same year prevented him from following the subject. After this initial favorable reception, Greenberger, Horne, and Zeilinger realized the subject deserved a better presentation than simply a conference paper.[5] They called Abner Shimony to join them in the writing of a more complete paper explaining the theorem and envisioning possible experiments. As we have seen, it was the second time Shimony was called by Horne and Zeilinger to further exploit and present their original ideas.

4. The Bell inequality

To understand the GHZ theorem, one might begin with the celebrated two-particle spin singlet state, with zero total angular momentum, introduced by David Bohm in his 1951 textbook, *Quantum theory*, and depicted in Fig. 1.

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[4] Interview with Michael Horne, by Joan Bromberg, 12 Sep 2002, AIP. Interview with Anton Zeilinger, by Olival Freire Jr., 30 June and 2 July 2014, AIP.

[5] In fact, there already was some competition around the most general proof of the theorem between Greenberger, Horne & Zeilinger on the one hand, and Clifton, Redhead & Butterfield on the other. This competition is recorded in the paper by Clifton et al. (1991), on a "note added in proof", on p. 182.

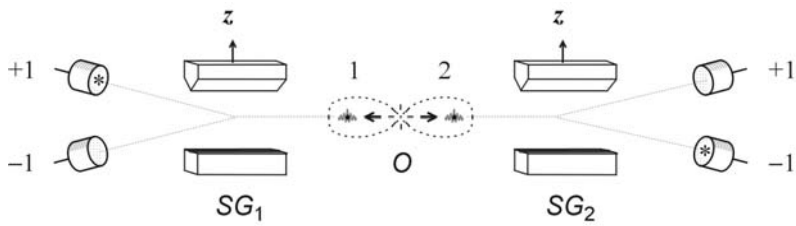

Figure 1. Two spin-½ particles emitted in an entangled state with perfect anticorrelation.

Stern-Gerlach (SG) magnets separate the beams in such a way that, for each particle, the probability of measuring each of the two possible outcomes is ½. The observable being measured is the *z*-component of spin, with the two possible eigenvalues (outcomes) conventionally chosen as +1 and –1, as shown in the figure. The singlet state is such that in ideal experimental situations, in which both SG magnets point in the same angle (the "superclassical" case, according to Greenberger et al. 1989), one measures perfect anticorrelation, meaning that if particle 1 is measured with the value of the spin component $I = +1$, particle 2 will necessarily be measured with the value $II = -1$, and vice-versa.

The experiment shown in Fig. 1 could be readily explained by a local realist theory: one could suppose that the spin particles are emitted with definite and opposite spin components in the z direction. What such a local realist theory cannot do is explain the fact that perfect anticorrelation would occur *for any angle* of the SG magnets, assuming that both magnets and pairs of detectors are quickly rotated in the same angle, right after the emission of the pair of particles.

This feature of rotational symmetry in the experimental outcomes implies that the quantum mechanical representation of the singlet state must have rotational symmetry. This appears in the following expression:

. (1)

$|+_2\rangle$

The notation indicates the eigenstate of the *z* component of spin in the positive direction, associated to eigenvalue +1, for particle 2. That the above expression is rotationally invariant may be verified by replacing each individual state by its representation in another basis (corresponding to the eigenstates for another angle of the SG magnets).

The proof that local realist theories cannot account for the predictions of quantum mechanics for pairs of entangled particles was derived by Bell in 1964, by means of an inequality that limits the predictions of any local realist theory, but may be violated by the quantum mechanical predictions. The version derived by Clauser, Horne, Shimony & Holt (1969) is:

(2)

Representing by $I(a)$ and $II(b)$ the results of the measurement for each pair of particles, for measurements performed with SG magnets adjusted respectively at angles *a* and *b*, then the correlation coefficient $c(a,b)$ is the mean of the products $I(a)\cdot II(b)$ averaged over all the pairs of particles.

Bell (1964) postulated the existence of a set of hidden variables λ that uniquely determine the values of the measurement outcomes. The general expression for each value (either 1 or –1) predicted by the realist hidden variables theory is $I(a,b,\lambda)$ and $II(a,b,\lambda)$. In order to impose the property of *locality* in these models, Bell dropped the dependence of a measurement outcome on the far away SG magnet; as a result of this factorizability condition, the values of outcomes are written simply as $I(a,\lambda)$ and $II(a,\lambda)$. With this, Bell was able to derive his inequality.

Quantum mechanics predicts that the correlation function for the singlet state is given by:

$$c_{\Psi_S}(a, b) = -\cos(a - b) \ . \tag{3}$$

One may find values for a, a', b, and b' for which the values obtained in (3) violate inequality (2).

One notes in this inequality that its experimental verification must involve four experimental runs, with four pairs of angles of the magnets. This introduces additional hypotheses of fair sampling in the experimental test of local realist theories.

If an experimental test could be done with a single setup, this would reduce the additional assumptions used to rule out local realist theories. The first derivation of a version of Bell's theorem without inequalities was done by Heywood & Redhead (1983), based on simplified versions of the Kochen-Specker theorem. A simpler and more testable approach was that of Greenberger, Horne & Zeilinger (1989), who derived a proof involving four spin-½ particles.

5. Scope of GHZ theorem

The four particle entangled state proposed by Greenberger et al. (1989) is written as follows:

. (4)

This state may be prepared in a similar way as the two-particle system of Fig. 1, assuming that each of the two entangled particles have spin 1, and that each of them decays into two spin-½ particles. This is represented in Figure 2.

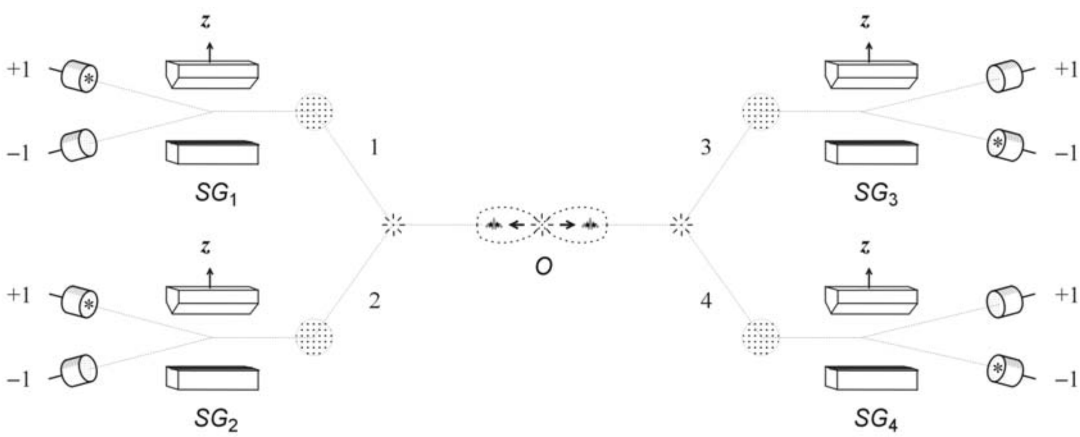

Figure 2. Experimental setup for four entangled spin-½ particles, proposed by GHZ, with every SG magnet pointing at the same angle *z*.

One notices again strict anticorrelation between measurements on the left side of the picture and on the right, as in Fig. 1. The situation imagined by GHZ to test local realist theories involves magnets at different angles, respectively a, b, c, d, so that the strict anticorrelation of the superclassical case will not occur.

The quantum mechanical correlation function associated to these measurements is similar to the two-particle case, and furnishes:

$$c_{\Psi ss}\,(a, b, c, d) = -\cos\,(a + b - c - d)\ . \tag{5}$$

One can now consider a case of strict anticorrelation ($c_{\Psi ss} = -1$), of which the setup of Fig. 2 is a special case, and also a case of strict correlation ($c_{\Psi ss} = 1$):

$$\begin{aligned} &\text{If } a + b - c - d = 0, \quad \text{then} \quad c_{\Psi ss} = -1. \\ &\text{If } a + b - c - d = \pi, \quad \text{then} \quad c_{\Psi ss} = 1\ . \end{aligned} \tag{6}$$

These situations have the property that if one measures the spin component of three particles, one knows for sure (in an ideal experimental situation) the value of the fourth one (in the special case depicted in Fig. 2, a single measurement is sufficient, as indicated in eq. 4). This is analogous, as pointed out by Greenberger et al. (1989). to what happens in the Einstein, Podolsky & Rosen argument, which is based on counterfactual strict anticorrelation measurements.

Let us now postulate the existence of hidden variables λ, that uniquely determine the measurement outcomes $I(a,\lambda)$, $II(b,\lambda)$, $III(c,\lambda)$, and $IV(d,\lambda)$ of the respective components of spin. As in the derivation of Bell's theorem, one has imposed the factorizability condition, which expresses the locality of the model. The value of each $I(a,\lambda)$, ..., is either 1 or –1.

Expressions (6) may be rewritten taking into account that, in the superclassical cases of strict correlation or anticorrelation, the product $I(a)\cdot II(b)\cdot III(c)\cdot IV(d)$ is equal to the correlation coefficient $c_{\Psi ss}$:

$$\text{If } a + b - c - d = 0, \text{ then } I(a,\lambda)\cdot II(b,\lambda)\cdot III(c,\lambda)\cdot IV(d,\lambda) = -1. \tag{7}$$
$$\text{If } a + b - c - d = \pi, \text{ then } I(a,\lambda)\cdot II(b,\lambda)\cdot III(c,\lambda)\cdot IV(d,\lambda) = 1.$$

GHZ were able to show that these two expressions lead to a contradiction. Starting with the first expression, one may derive four special cases, where ϕ is an arbitrary angle:

$$\begin{aligned} I(0,\lambda)\cdot II(0,\lambda)\cdot III(0,\lambda)\cdot IV(0,\lambda) &= -1, \\ I(\phi,\lambda)\cdot II(0,\lambda)\cdot III(\phi,\lambda)\cdot IV(0,\lambda) &= -1, \\ I(\phi,\lambda)\cdot II(0,\lambda)\cdot III(0,\lambda)\cdot IV(\phi,\lambda) &= -1, \\ I(2\phi,\lambda)\cdot II(0,\lambda)\cdot III(\phi,\lambda)\cdot IV(\phi,\lambda) &= -1. \end{aligned} \tag{8}$$

Multiplying the first three equations of (8), and recalling that the values *I*, *II*, *III*, and *IV* are either 1 or –1, one obtains:

$$I(0,\lambda)\cdot II(0,\lambda)\cdot III(\phi,\lambda)\cdot IV(\phi,\lambda) = -1. \tag{9}$$

Comparing this with the last equation of (8), one concludes that $I(2\phi,\lambda) = I(0,\lambda)$, i.e., $I(\psi,\lambda)$ has a constant value for all ψ. That would be very strange, for if $\psi = \pi$, one expects that the value of *I* would have a negative sign in relation to the value for $\psi=0$.

For this to constitute a mathematical contradiction, one may take the second expression of (7) and derive the following special case:

$$I(\phi + \pi,\lambda)\cdot II(0,\lambda)\cdot III(\phi,\lambda)\cdot IV(0,\lambda) = 1. \tag{10}$$

Together with the second of equations (8), one concludes that $I(\phi + \pi,\lambda) = -I(\phi,\lambda)$, which contradicts the previous assertion that $I(\psi,\lambda)$ would have a constant value for all ψ.

Therefore, the assumption that quantum mechanics can be formulated as a realist hidden variable theory leads to a contradiction, without the use of inequalities!

6. Epilogue: GHZ experiments

The production of GHZ states in laboratories was neither an easy job nor a quick achievement. It lasted almost a decade. In fact, from 1990 on, Bell's reaction, in particular, motivated Zeilinger to immediately think about taking this theorem to the laboratory benches. The road to experiments however, was dependent on both conceptual and experimental advances. Zeilinger considers it the most challenging experiment he had ever carried out. The quest for the required expertise, however, brought important preliminary results and spinoffs, such as the concept of "entanglement swapping" and the experiment on teleportation (Żukowski et al., 1993; Bouwmeester et al., 1997).[6] The 20th century closed with Zeilinger successfully obtaining the experimental production of GHZ states which are in agreement with quantum theory and in disagreement with local realistic theories (Bouwmeester et al., 1999).

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[6] Anton Zeilinger, interviewed by Olival Freire Jr., ibid.

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