A detailed theory of quantum premeasurement dynamics is presented in which a unitary composite-system operator that contains the relevant object-measuring-instrument interaction brings about the final premeasurement state. It does not include collapse, and it does not consider the environment. It is assumed that a discrete degenerate or non-degenerate observable is measured. Premeasurement is defined by the calibration condition, which requires that every initially statistically sharp value of the measured observable has to be detected with statistical certainty by the measuring instrument. The entire theory is derived as a logical consequence of this definition using the standard quantum formalism. The study has a comprehensive coverage, hence the article is actually a topical review. Connection is made with results of other authors, particularly with basic works on premeasurement. The article is a conceptual review, not a historical one.

General exact premeasurement is defined in 7 equivalent ways. A step is taken towards complete measurement (that includes collapse). Nondemolition premeasurement, defined by requiring preservation of any sharp value of the measured observable, is characterized in 10 equivalent ways. Overmeasurement, i.e., a process in which the observable is measured on account of being a function of a finer observable that is actually measured, is discussed. Disentangled premeasurement, in which, by definition, to each result corresponds only one pointer-observable state in the final composite-system state, is investigated. Ideal premeasurement, a special case of both nondemolition premeasurement and disentangled premeasurement, is defined, and its most important properties are discussed. Finally, disentangled and entangled premeasurements, in conjunction with nondemolition or demolition premeasurements, are used for classification of all premeasurements.

In concluding remarks the omitted aspects of the intricate topic of quantum measurement theory are shortly enumerated.

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I. INTRODUCTION

To cast a quick glance at history, one can say that the quantum-mechanical theory of premeasurement began in the last chapter of von Neumann's celebrated book (von Neumann, 1955). Since then numerous authors gave contributions, but none as many, so it seems to me, as Peka J. Lahti with various coauthors and by himself. (See his review Lahti, 1993. Some of his articles and those of others will be cited when they overlap with the claims made and proved in the present article.) The efforts of Lahti and coauthors seem to have culminated in the book by Busch, Lahti, and Mittelstaedt, 1996, which, no doubt, completed the mentioned work of von Neumann in a masterful fashion.

Nevertheless, a large informational gap was left in the wake of this awesome work. The gap is between its highbrow mathematical language and wide generality adopted in the book on the one hand, and the concepts and symbols that are standard in most quantum-mechanical textbooks and articles on the other.

Besides, there is the complementarity between the whole and the part well-known even in everyday experience: viewing the wood, you don't see the trees; watching the trees, the wood is out of sight. Comparing Busch, Lahti, and Mittelstaedt 1996 to a magnificent 'wood', the 'trees' would be the generalized observables (so-called positive-operator valued or POV measures; see concluding remark IV in section X), and the ordinary observables could be likened to 'bushes'.

If I am allowed to follow this metaphor a step further, I think that most of us quantum physicists 'live' in the 'bushes'. I think that a balanced approach between the general and the special is desirable, in which one confines
one to premeasurement of ordinary observables with purely discrete spectra and with arbitrarily degenerate or non-degenerate eigenvalues. As to the formalism, it is good to stick to the widely used practical Dirac notation. It follows von Neumann’s representation-independent (or abstract) treatment, which is best suited for fundamental quantum-mechanical investigations. Besides, unitary evolution operators are made use of. These are the confines of the present topical review, which is written mostly in the way of a research article because almost everything that is claimed is derived from the basic simple definition of premeasurement.

Measurement theory investigates composite systems consisting of the measured object, we will denote it as subsystem A, and the measuring instrument, subsystem B. The basic tool for the treatment of subsystems is the use of partial traces. These have three very practical accompanying rules (much utilized in the present article).

\[ \text{tr}_B(Y_A X_{AB}) = Y_A \text{tr}_B X_{AB} \equiv Z^{(1)}_A, \quad (1a) \]
\[ \text{tr}_B(X_{AB} Y_A) = (\text{tr}_B X_{AB}) Y_A \equiv Z^{(2)}_A, \quad (1b) \]

where \( Y_A = Y_A \otimes I_B \) (\( I_B \) being the identity operator for subsystem B) and \( X_{AB} \) are arbitrary subsystem and composite-system operators respectively, and the operators \( Z^{(i)}_A \ i = 1, 2 \), acting (in general nontrivially) in the subsystem state space opposite to that of the partial trace, are defined by the rest in relations (1a,b). Further, one has

\[ \text{tr}_B(Y_B X_{AB}) = \text{tr}_B(X_{AB} Y_B) \equiv Z^{(3)}_A. \quad (1c) \]

We will refer to (1c) as to ‘under-the-partial-trace commutativity’.

These general rules are easily proved by straightforward evaluation of both sides in an arbitrary pair of complete orthonormal bases \( \{|k\}_A : \forall k \}, \{|n\}_B : \forall n \} \). The use of partial traces with the rules (1a-c) is wider than measurement theory; it can be applied in any quantum-mechanical treatment of composite systems.

**Literature** To my knowledge, the partial trace was introduced by von Neumann (1955, section 2 of Chapter VI).

The basic aim of this article is to present general premeasurement and its important special case non-demolition premeasurement in as many details as possible. The above explained balance has enabled the present author to derive 7 equivalent definitions for general premeasurement, and 10 equivalent definitions for non-demolition premeasurement. The rest is a conceptual framework.

In premeasurement the separate results are not selected out, which would require theoretically some kind of collapse.

**Literature** In Busch, Lahti, and Mittelstaedt 1996 the term "premeasurement" is used. Complete measurement is called "premeasurement" with "objectification". The latter term is a synonym of "collapse".

Complete measurement, which by definition ends in one definite result, consists of two parts: In the first, the measured object interacts in a specific way with the measuring instrument, and specific quantum correlations are created. In the second part, the correlations are 'read' as information about the object on part of the measuring instrument, which thus becomes a 'subject' (it is here a 'technical' term). In the second part, as we know from experience, collapse to one definite result takes place. In the present article only the first part is investigated by applying unitary evolution operators, and by treating the measuring instruments as quantum-mechanical systems.

One should keep in mind that unitary quantum mechanics, opponents call it sometimes "bare quantum mechanics", is actually the textbook quantum mechanics, in which all dynamical changes are expressed via the famous Schrödinger equation.

Collapse has been for a long time paradoxical from the point of view of unitary dynamics, because it cannot be derived from the latter in a consistent and expected way (von Neumann, 1955, last section VI.3). It can be done in an unexpected way called relative-state or Everettian theory Everett 1957, Everett 1973 (cf remark II in the concluding remarks).

By "measurement" one usually means complete measurement in the literature. This term will be used in the present article when one has both or either of premeasurement and complete measurement in mind.

The present article will use the following conventions. Some physical notions and their mathematical representatives will be used interchangeably throughout, like 'pure state' and 'state vector' (vector of norm one); 'state', 'density operator', and 'state operator'; 'observable' and 'Hermitian operator'; 'compatible' observables and 'commuting' Hermitian operators; event' and 'projector'. Complete orthonormal bases in a given state space will be called simply 'bases'. Vectors of norm that is not necessarily one will be written overlined; non-overlined vectors will always be of norm one.

Tensor products of vectors will mostly be written without a multiplication sign, but sometimes, for emphasis, such a sign will be utilized, e. g., \( |\phi\rangle_A^\dagger \otimes |\phi\rangle_B^\dagger = |\phi\rangle_A^\dagger \otimes |\phi\rangle_B^\dagger \) (as will be used below). Whenever possible, strings of symbols will begin by numbers that will be followed by the sign "×". We use
the convention that if in a string of entities the first is zero, the rest need not be defined; the string is by definition zero.

It should be noted that a "necessary and sufficient condition" is another definition. Occasionally also the terms "characterization", "characteristic properties", "criterion", and "condition" will be used as further synonyms of "definition".

In order to avoid overburdening the article with mathematics, mathematical presentation in terms of lemmata, theorems etc. will be avoided and boldface "claims", strictly identified via the numbers of the corresponding relations, will be used instead. This seems more appropriate for a physical text. Besides, it is easier for the reader to look up a given relation than to find a lemma or a theorem etc. because the former are all consecutively enumerated, whereas in the latter case the lemmata, theorems etc. are usually each separately consecutively enumerated.

Both the proofs and the remarks on literature are written in italics to enable the reader to skip them easily in a first reading.

II. GENERAL PREMEASUREMENT

Let subsystem A be the object of measurement, and let

$$O_A = \sum_k o_k E^k_A, \quad k \neq k' \Rightarrow o_k \neq o_{k'} \quad (2a)$$

be the unique spectral form of the measured discrete observable, which may have an infinite purely discrete spectrum and each eigenvalue may have an arbitrary (finite or infinite) degeneracy or lack of it. By 'uniqueness' is meant the non-repetition of the eigenvalues \(\{o_k : \forall k\}\) in (2a). Henceforth, we always mean by 'spectral form' the unique one unless otherwise stated. The symbol \(\Rightarrow\) denotes logical implication.

Also the completeness relation \(\sum_k E^k_A = I_A\) is satisfied.

Let, further, subsystem B be the measuring instrument equipped with a so-called pointer observable

$$P_B = \sum_k p_k F^k_B, \quad (2b)$$

in its spectral form. Also (2b) is accompanied by the corresponding completeness relation \(\sum_k F^k_B = I_B\). The eigen-projectors \(F^k_B\) of the pointer observable will be called 'pointer positions' in the present article. (The term usually applies rather to the eigenvalues \(p_k\) in the literature, but they will be seen to play an inferior role.)

**Literature** We make \(O_A\) and \(P_B\) the basic 'players' in the theory following von Neumann; cf von Neumann 1955, end of the third page of section 3. of chapter VI.

The premeasurement interaction establishes entanglement between object and measuring instrument in a specific way in the final composite premeasurement state. It is investigated in what follows. The unitary operator incorporating the premeasurement interaction and mapping the initial composite-system state vector \(|\phi\rangle_A^1 \otimes |\phi\rangle_B^1\) into the final state (at the end of premeasurement interaction) we denote by \(U_{AB}\):

$$\forall |\phi\rangle_A^1 \in \mathcal{H}_A : \quad \Phi_{AB}^f \equiv U_{AB} \left( |\phi\rangle_A^1 \otimes |\phi\rangle_B^1 \right). \quad (3)$$

Here \(|\phi\rangle_A^1\) is an arbitrary initial state vector of the measured system \(A\), \(\mathcal{H}_A\) is the state space (complex separable Hilbert space) of the object, and \(|\phi\rangle_B^1\) is the initial or ready-to-measure state vector of the instrument. The superscripts refer to the initial and the final state respectively.

The triple \(|\phi\rangle_B^1, P_B, U_{AB}\) is sometimes called the "measuring instrument" in the formalism of unitary measurement theory.

The degeneracies (or multiplicities) of the eigenvalues \(o_k\) of the measured observable may be arbitrary. Also the ranges \(\mathcal{R}(F^k_B)\) of the 'pointer positions' \(F^k_B\) may be degenerate with any degeneracy including the possibility of all being non-degenerate, when one can write \(P_B = \sum_k p_k |k\rangle_B \langle k|_B\) (cf (2b)).

**Literature** For the last mentioned entirely non-degenerate choice of the pointer positions, Busch, Lahlé, and Mittelstaedt (1996) say that "it is minimal in the sense that it is just sufficient to distinguish the eigenvalues of" the measured observable (2a) (cf subsection III.2.3 in their book, where a different notation is used).

Premeasurement is defined by the so-called calibration condition: If the initial state of the object has a sharp (or definite) value of the measured observable, then the final composite-system state has the corresponding sharp value of the pointer observable. 'Corresponding' we write as 'having the same index value' \(k\).

It is firmly established that the quantum-mechanical relations have a statistical meaning and are tested on ensembles of equally prepared systems. In particular, the precise statistical form of the calibration condition is expressed in terms of the standard probability formulae: For any fixed value \(k\) of the index \(k\), one has

$$\langle \phi\rangle_A^1 E^k_A |\phi\rangle_A^1 = 1 \Rightarrow \langle \Phi_{AB}^f F^k_B |\Phi_{AB}^f \rangle^f = 1, \quad (4)$$

and the final state \(|\Phi_{AB}^f\rangle\) is given by (3). Equivalent definitions of the calibration condition that are more operational are derived below.
The obvious physical meaning of the calibration condition (4) is that a (statistically) sharp value of the measured observable must be (statistically) sharply detected by the measuring instrument. The calibration condition is obviously a necessary requirement for pre-measurement. Following Busch, Lahti, and Mittelstaedt 1966 for our restricted choice of observable, we take the calibration condition as the definition, i.e., as a necessary and sufficient condition, for general exact pre-measurement.

**Literature** The calibration condition is given in Busch, Lahti, and Mittelstaedt 1996, subsection III.2.3.

The recent ontic breakthrough (Pussey, Barrett, and Rudolph 2012 etc.; cf Herbut 2014a and Leifer 2014) makes it plausible that the state vector is a property of the individual system. Then it is desirable to understand the calibration condition as a primarily individual-system requirement. This means that it may go beyond statistical (and ensemblewise) meaning (relations (4)) towards the individual systems in the following sense. If the calibration condition were a purely statistical requirement, then it would allow not to be true on some individual systems in any finite ensemble, say on N of them. But the relative number N'/N of such exceptions is required to tend to zero as N, the number of systems in the ensemble, tends to infinity. If the calibration condition is an individual-system requirement, then it is never allowed to be violated by any individual system in pre-measurement.

Nevertheless, the purely statistical notion (4) for pre-measurement is more suited in view of the fact that all quantum-mechanical formulae have, actually, a statistical meaning. One must also be aware that there exist strictly ensemble measurements that cannot be given individual-system meaning. A well-known elementary example is two-slit interference. (When the individual photon hits the second screen, this does not tell much about it. Only an ensemble of photons can reveal interference.) Such measurements are outside the scope of this investigation.

To derive an equivalent, more practical, form of (4), we need a useful, general, known, but perhaps not well known, auxiliary algebraic certainty claim.

An event E is certain, i.e., has probability one, in a pure state |ψ⟩ if and only if the latter is invariant under the action of E:

⟨ψ|E|ψ⟩ = 1 ⇔ E|ψ⟩ = |ψ⟩.  

(The symbol "⇔" denotes logical implication in both directions.) Proof is given, for the reader's convenience, in the Appendix.

Equivalence (5) makes it obvious that the calibration condition can be equivalently defined by the more operational invariance form of the calibration condition, which is:

|φ⟩_A^k = E^k_A |φ⟩_A \Rightarrow |Φ⟩_{AB} = F^k_B |Φ⟩_{AB}. \quad (6)

**Proof** is a straightforward consequence of the form (6) of the calibration condition and the linearity and the continuity of all operations involved. Let |φ⟩_A^k = E^k_A |φ⟩_A^k and, let |φ⟩_B and |χ⟩_B^k be two initial states of the measuring instrument for which the calibration condition is valid. Then, for any complex numbers \(\alpha, \beta\) satisfying \(|\alpha|^2 + |\beta|^2 = 1\), (6) implies

\[ U_{AB} \left[ |φ⟩_A \otimes (\alpha |φ⟩_B + \beta |χ⟩_B) \right] = \alpha F^k_B U_{AB} \left[ |φ⟩_A \otimes |φ⟩_B \right] + \beta F^k_B U_{AB} \left[ |φ⟩_A \otimes |χ⟩_B \right]. \]

Thus, the calibration condition is valid also for \(\alpha |φ⟩_B + \beta |χ⟩_B\).

Analogously, one proves preservation under the limit process: Let \(\{|ψ⟩_B\}^n_1 : n = 1, 2, \ldots, \infty\) be a sequence of states observing the calibration condition that converge to \(ψ⟩_B\), then the calibration condition is preserved under the limit due to the continuity of both the unitary evolution operator and the projector \(F^k_B\).

The subspace \(S^k_B\) can be one-dimensional. As an alternative, one can define each pre-measurement with a fixed initial state \(ψ⟩_B\) disregarding the dimension of \(S^k_B\). This will be done throughout in this article in order to avoid overburdening the presentation.

Now we state and prove the claim of the dynamical necessary and sufficient condition for pre-measurement.

One has premeasurement if and only if

\[ \forall |φ⟩_A, \forall k : \left( F^k_B U_{AB} \right) |φ⟩_A = U_{AB} E^k_A |φ⟩_A |φ⟩_B \]

\[ \left( F^k_B U_{AB} \right) |φ⟩_A |φ⟩_B \]

\[ (8) \]
is valid.

One proves necessity as follows. For each $k$ value the completeness relation $\sum_k E_A^{k} = I_A$, repeated use of the calibration condition in its invariance form (6), and finally orthogonality and idempotency of the $F_B^k$ projectors enable one to write

$$F_B^k U_{AB} \left( |\phi_A^i> |\phi_B^j> \right) = \sum_{k'} \|E_A^{k'}|\phi_A^i>\| \times F_B^k U_{AB} \left( \left( E_A^{k'} |\phi_A^i> / \|E_A^{k'}|\phi_A^i>\| \right) |\phi_B^j> \right) = \sum_{k'} |E_A^{k'}|\phi_A^i>\| \times F_B^k F_B^{k'} U_{AB} \left( \left( E_A^{k'} |\phi_A^i> / \|E_A^{k'}|\phi_A^i>\| \right) |\phi_B^j> \right) = \|E_A^i|\phi_A^i>\| \times F_B^k U_{AB} \left( \left( E_A^i |\phi_A^i>/ \|E_A^i|\phi_A^i>\| \right) |\phi_B^j> \right) = \|E_A^i|\phi_A^i>\| \times U_{AB} \left( \left( E_A^i |\phi_A^i>/ \|E_A^i|\phi_A^i>\| \right) |\phi_B^j> \right).$$

After cancellation of $\|E_A^i|\phi_A^i>\|$, the claimed relation (8) follows.

Note that the argument covers both the case $\|E_A^i|\phi_A^i>\| > 0$ and $\|E_A^i|\phi_A^i>\| = 0$ due to the convention that, when the first factor in a term is zero, then, by definition, the entire term is zero. Namely, in the latter case, it has just been shown that if $E_A^i |\phi_A^i> = 0$, then also $F_B^k (\Phi^i_A) = 0$, i.e., (8) is valid.

To prove sufficiency, let (8) be valid, and let $|\phi_A^i> = E_A^i |\phi_A^i>\|$ be satisfied. Then, one has

$$\left( U_{AB} E_A^i \right) \left( |\phi_A^i> |\phi_B^j> \right) = \left( F_B^k U_{AB} \right) \left( |\phi_A^i> |\phi_B^j> \right).$$

One can here omit $E_A^i$ due to the above assumption, and thus the calibration condition in its invariance form (6) $|\Phi^i_{AB} = F_B^k |\Phi^i_{AB} \rangle \rangle$ is obtained. Hence, we do have premeasurement.

**Literature** The dynamical definition of general premeasurement (8) is introduced and proved in Herbut 2014c.

In an attempt to comprehend the meaning of the dynamical criterion (8) intuitively, one may realize that complete measurement, which is beyond this study, would collapse the final premeasurement state $|\Phi^i_{AB}>$, given by (3) into one of its sharp pointer-position projections $F_B^k |\Phi^i_{AB}> \langle F_B^k |\Phi^i_{AB}>\|$, and $E_A^i \langle|\Phi^i_{AB}> \langle F_B^k |\Phi^i_{AB}>\|$, is the corresponding collapsed form of the initial state. Then (8) says that it applies to the same whether the collapse takes place at the beginning and then evolution sets in or at the end after the evolution.

Important consequences of the dynamical definition (8) of premeasurement apply to a connection with complete measurement. These consequences are discussed in the next section.

To express the **claim of the basis-dynamical characterization** of general premeasurement, we take an arbitrary basis $\{ |k,q_0>, A \} \forall q_0$ in each eigen-subspace $\mathcal{R}(E_A^i)$ of the measured observable $O_A$ ($= \sum_k q_0 E_A^k$). Then, the final state (3) is that of a premeasurement if and only if

$$\forall k,q_0: \quad U_{AB} \left( |k,q_0>, A |\phi_B^j> \right) \in \mathcal{R}(I_A \otimes F_B^k). \quad (9a)$$

**Proof of necessity** is obvious, and so is that of **sufficiency** if one has in mind that an initial state $|\phi_A^i>$ has a sharp value $q_0$ of $O_A$ if and only if it can be expanded as $|\phi_A^i> = \sum q_0 \left( k,q_0 |A |\phi_A^i> \right)$ (cf. the calibration condition (6)).

Since the power of the set $\{ |k,q_0>, A |\phi_B^j> \} \forall q_0$ equals the dimension of $\mathcal{R}(E_A^i)$ and since $dim(\mathcal{R}(E_A^i)) \leq dim(H_A) \leq dim(H_A) \times dim(\mathcal{R}(F_B^k))$, a unitary operator $U_{AB}$ satisfying the basis-dynamical condition can always be constructed. Thus, all discrete observables are, in principle, exactly measurable. (At least there is no algebraic obstacle for this; cf concluding remark VI in section X.)

**Literature** In the book Busch, Lahti, and Mittelstaedt 1996 this is proved in Theorem 2.3.1 with the assumption of a pointer observable that is discrete and nondegenerate (\forall k: \quad \{ k \in \mathcal{R}(F_B^k) \} = 1).

It is obvious that the basis-dynamical condition has its equivalent form in the **subspace-dynamical condition** (like the other face of the coin): The final state $|\Phi^i_{AB}>$ is that of a premeasurement if and only if each eigen-subspace $\mathcal{R}(E_A^i)$ of the measured observable is taken into the corresponding eigen-subspace $\mathcal{R}(I_A \otimes F_B^k)$ by the unitary evolution operator $U_{AB}$. In terms of a system of formulae the characterization has the precise form

$$\forall k: \quad U_{AB} [\mathcal{R}(E_A^i) \otimes \mathcal{R}(|\phi_B^j>_A)] \subseteq \mathcal{R}(I_A \otimes F_B^k), \quad (9b)$$

where by a tensor (or direct) product of subspaces is meant the span of the set of all tensor products of an element from the first and an element from the second factor space.

Our next **claim of the probability reproducibility condition** as a necessary and sufficient condition for premeasurement reads: The **probability of a result $E_A^i$ predicted by any initial state $|\phi_A^i>$ of the object equals the probability of the corresponding pointer position $F_B^k$ in the final composite-system state:**

$$\forall |\phi_A^i>, \forall k: \quad \langle \phi_A^i | E_A^i |\phi_A^i> = \langle \Phi^i_{AB} | F_B^k |\Phi^i_{AB}>. \quad (10)$$
One should note that one way to put the calibration condition is to claim that every Kronecker distribution of probability in the sense of LHS(10) equals the corresponding Kronecker probability distribution in the sense of RHS(10). Obviously, the probability reproducibility condition is valid only if so is the calibration condition. Though the probability reproducibility condition apparently requires much more than the calibration condition, perhaps surprisingly, the former is valid if and only if so is the latter (both in the statistical sense).

**Literature** This is a known fact, cf. Busch, Lahti, and Mittelstaedt 1996, subsection III.1.2.

To **prove** that the calibration condition implies the probability reproducibility condition, we argue as follows. On account of (8), (3), and the idempotency of the projectors, RHS(10) equals

$$\langle \phi | A (E^k_A U^i_B U_{AB}^j)(U_{AB} E^i_A) | \phi \rangle_A = LHS(10).$$

Intuitively, one may remark that the probability reproducibility condition (10) displays the kind of information that is transmitted from object to measuring instrument when measurement entanglement in the final state is established.

Our next **claim** is another necessary and sufficient condition for premeasurement.

The strong invariance form of the calibration condition reads: The calibration condition is valid if and only if the premeasurement entities \{U_{AB}, P_B, |\phi^i_B\rangle\} lead to a final state $|\Phi^f_{AB}\rangle$ so that

$$|\phi^i_A\rangle = E^k_A |\phi^i_A\rangle \iff |\Phi^f_{AB}\rangle = F^k_B |\Phi^f_{AB}\rangle$$

(11) holds true.

Claim (11) **follows** immediately from the probability reproducibility condition (10) because, according to the latter,

$$\langle \phi | A E^k_A | \phi \rangle_A = 1 = \langle \Phi^f_{AB} F^k_B | \Phi^f_{AB}\rangle$$

(cf (4), (5) and (6)).

The strong invariance form of the calibration condition implies that the calibration condition (6) is essentially satisfied also in the time-reversed situation. Hence, the strong invariance form (11) of the calibration condition implies a kind of time reversal symmetry in premeasurement: If we slightly generalize the premeasurement concept allowing the initial object+instrument system to be correlated (due to previous interaction that was completely independent of the premeasurement), and we apply time reversal to the premeasurement process, then $|\Psi^f_{AB}\rangle$ becomes the initial state, the instrument, subsystem B, is then the measured object, the former pointer observable $P_B \left( = \sum_k c_k E^k_B \right)$ is the measured observable $O_A \left( = \sum_k a_k E^k_A \right)$ is now the pointer observable with its eigen-projectors $\{E^k_A : \forall k\}$ as the 'pointer positions'.

Now $|\phi^i_A\rangle$ reproduces the relevant information contained in $|\Psi^f_{AB}\rangle$ in terms of the probability reproducibility condition

$$\forall |\Phi^f_{AB}\rangle, \forall k : \langle \phi^i_A | E^k_A | \phi^i_A \rangle = \langle \Psi^f_{AB} F^k_B | \Psi^f_{AB}\rangle.$$ (12)

**Literature** Peres gave a similar discussion in Peres 1974.

Finally, we turn to the canonical subsystem-basis expansion criterion for premeasurement. The **claim** reads: an object+(measuring instrument) state $|\Phi^f_{AB}\rangle \equiv U_{AB}(|\phi^i_A\rangle + \phi^i_B)$ is the final state of premeasurement, i.e., $|\Phi^f_{AB}\rangle = |\Phi^f_{AB}\rangle$, if it, expanded in some eigen-basis $\{|k, s_k\}_B : \forall k, s_k\}$ of the pointer observable $P_B \left( = \sum_k p_k F^k_B \right)$, in

$$|\Phi^f_{AB}\rangle = \sum_k \sum_{s_k} |k, s_k\rangle_A |k, s_k\rangle_B.$$ (13a)

The 'expansion coefficients' (elements of $H_A$) $\{|k, s_k\}_A : \forall k, s_k\}$ satisfy the following relations:

$$\forall k : \sum_{s_k} ||k, s_k\rangle_A||^2 = \langle \phi^i_A | E^k_A | \phi^i_A \rangle.$$ (13b)

One is dealing with the final state of an premeasurement **only if** relations (13a,b) are valid for every eigenbasis of the pointer observable.

**Proof** of the claim follows immediately from the fact that (13a) and (13b) together are an equivalent formulation of the probability reproducibility condition. Namely, $\forall k :$ \[LHS(13b) = ||F^k_B | \Phi^f_{AB}\rangle||^2 = \langle \Phi^f_{AB} F^k_B | \Phi^f_{AB}\rangle = RHS(10).\]

Note that the 'expansion coefficients' in (13a) are, apart from (13b), arbitrary vectors in $H_A$. This is what makes the premeasurement that we investigate in this section general. Kinds of premeasurement will be specified in the special cases studied further below.

**Literature** A somewhat simplified form of the above characterization of premeasurement in terms of the form of the final state was obtained by Lahti (1990, relation (12) there).

**III. TOWARDS COMPLETE MEASUREMENT**

To begin with, we specify the explicit expanded form of the final premeasurement state (3) in terms of final
complete-measurement states $F^f_B | \Phi^f_{AB} \rangle / \| F^f_B | \Phi^f_{AB} \rangle $:

$$| \Phi^f_{AB} \rangle = \sum_k (\langle \phi^1_A | E^k_A | \phi^1_A \rangle)^{1/2} \times \left( F^f_B | \Phi^f_{AB} \rangle / \| F^f_B | \Phi^f_{AB} \rangle \right)$$

valid for every $| \phi^1_A \rangle$ (cf (3)). Naturally, it follows from $\| F^k_B | \Phi^f_{AB} \rangle = \left( \langle \Phi^f_{AB} | F^k_B | \Phi^f_{AB} \rangle \right)^{1/2}$ and the probability reproducing condition (10).

Expansion (14) displays a connection between premeasurement and complete measurement.

The concept of complete measurement utilized in (14) is without overmeasurement (cf section V). In other words, each value of the measured observable is measured independently (more in section V). From this state by a sub-projector of $F^k_B$ is obtained because this process ends in the state $F^k_B | \Phi^f_{AB} \rangle$.

Further, we refer to the corresponding final vector $| \Phi^f_{AB} \rangle$ of the object state. Further, we refer to the corresponding final vector $F^k_B | \Phi^f_{AB} \rangle$ in the expansion $| \Phi^f_{AB} \rangle = \sum_{k'} F^k_B | \Phi^f_{AB} \rangle$, due to $\sum_{k'} F^k_B = I_B$, as the $k$-th complete-measurement final component. (Note that both are not of norm one in general.)

The initial and final components are closely connected. Namely, the dynamical condition (8) implies that the $k$-th initial component contributes to the $k$-th complete-measurement final component in the unitary evolution of premeasurement:

$$\forall | \phi^1_A \rangle, \forall k : F^k_B | \Phi^f_{AB} \rangle = U_{AB} \left( E^k_A | \phi^1_A \rangle \otimes | \phi^1_B \rangle \right).$$

Relation (15) is actually (8) rewritten.

Claim (15) is relevant for complete measurement of the observable $O_A$ in which the result $o_k$ is obtained because this process ends in the state $F^k_B | \Phi^f_{AB} \rangle / \| F^k_B | \Phi^f_{AB} \rangle$ or in a normalized projection of this state by a sub-projector of $F^k_B$ in case of overmeasurement.

We refer to the initial and the final $k$-th components together as to the $k$-th branch (or channel) of premeasurement. This is a concept that applies to the entire premeasurement process, not to any fixed moment $t$, $t^i \leq t \leq t^f$ in it.

What relation (15) 'tells us' can be put in intuitive terms as follows: The measurement process takes place within each branch separately, independently of the rest of the branches.

IV. NONDEMOLITION PREMEASUREMENT

Nondemolition (synonyms: predictive, repeatable, first kind) premeasurement is defined as a premeasurement satisfying the additional requirement that, if the measured observable $O_A \left( = \sum_k o_k E^k_A \right)$ has a sharp value in the initial state, then, in the final state, the sharp value is preserved (it is not demolished).

In the statistical language of quantum-mechanics the additional nondemolition requirement reads:

$$\langle \phi^1_A | E^k_A | \phi^1_A \rangle = 1 \Rightarrow \langle \Phi^f_{AB} | E^k_A | \Phi^f_{AB} \rangle = 1. \quad (16a)$$

This together with the statistical form of the calibration condition (4) is the extended statistical calibration condition definition of nondemolition premeasurement.

Using the more practical form (5) of certainty, the nondemolition additional condition (16a) can be given the more practical equivalent invariance form

$$| \phi^1_A \rangle = E^k_A | \phi^1_A \rangle \Rightarrow | \Phi^f_{AB} \rangle = E^k_A | \Phi^f_{AB} \rangle. \quad (16b)$$

If joined to the invariance form of the calibration condition (6) of general premeasurement, then we have the extended invariance form of the definition of nondemolition premeasurement.

We have also the additional strong invariance requirement

$$| \phi^1_A \rangle = E^k_A | \phi^1_A \rangle \Leftrightarrow | \Phi^f_{AB} \rangle = E^k_A | \Phi^f_{AB} \rangle. \quad (17)$$

which together with (11) is the extended strong invariance definition of nondemolition premeasurement.

The equivalent definitions of nondemolition premeasurement are presented, as far as possible, in an order parallelling that of the definitions of general premeasurement. For this reason, (17) is given here without proof. Proof is supplied below; (17) is a consequence of (20).

Since requirement (16b) of nondemolition of the measured eigenvalue of $O_A$, joined to the calibration condition, makes the premeasurement evolution operator $U_{AB}$ more specified, it is to be expected that it stands in some additional relation to the eigen-projectors $E^k_A$ (with respect to that in the dynamical relation (8)). Indeed, so it is. The following additional system of dynamical necessary and sufficient conditions for a general premeasurement to be a nondemolition one is valid.

The claim of the additional dynamical condition goes as follows. A premeasurement is a nondemolition
one if and only if the evolution operator \( U_{AB} \) commutes with each eigen-projector \( E_k^B \) of the measured observable \( O_A \) when acting on \( |\phi_A^i \rangle \). In terms of formulae, a premeasurement is a nondemolition premeasurement if and only if

\[
\forall |\phi_A^i \rangle, \forall k : \left( U_{AB} E_k^B \right) (|\phi_A^i \rangle |\phi_B^i \rangle) = \left( E_k^B U_{AB} \right) (|\phi_A^i \rangle |\phi_B^i \rangle)
\]

is satisfied.

To prove necessity, let \( \{ (k, q_k)_A : \forall k, \forall q_k \} \) be an eigenbasis of the measured observable:

\[
\forall k, k', q_k, q_{k'} : E_k^A |k', q_{k'} \rangle_A = \delta_{k,k'} |k, q_k \rangle_A ,
\]

and let \( |\phi_B^i \rangle \) observe the calibration condition and the nondemolition requirements (16b) with all pure initial states of the object. Then the eigenvalue relations and the nondemolition requirement (16b) imply

\[
\forall k, k', q_k, q_{k'} : \left( U_{AB} E_k^B \right) (|k, q_k \rangle_A |\phi_B^i \rangle) = \delta_{k,k'} \times U_{AB} \left( |k, q_k \rangle_A |\phi_B^i \rangle \right) .
\]

On the other hand, using (16b) again, one has

\[
\left( E_k^B U_{AB} \right) (|k', q_{k'} \rangle_A |\phi_B^i \rangle) = E_k^B E_k^B \left( U_{AB} |k', q_{k'} \rangle_A |\phi_B^i \rangle \right) = \delta_{k,k'} E_k^B U_{AB} (|k', q_{k'} \rangle_A |\phi_B^i \rangle) .
\]

Since the right-hand sides are equal (if \( k = k' \), and both are zero otherwise), so are the left-hand sides.

**Sufficiency** is proved in the following way. Let \( |\phi_A^i \rangle = E_k^A |\phi_A^i \rangle \) and let condition (18) be valid. Then

\[
U_{AB} (|\phi_A^i \rangle |\phi_B^i \rangle) = E_k^B U_{AB} (|\phi_A^i \rangle |\phi_B^i \rangle) .
\]

Hence, on account of definition (3), also \( |\Phi_{AB}^k \rangle = E_k^B |\Phi_{AB}^k \rangle \) , i.e., relation (16b) is satisfied.

The dynamical condition (8) and (18) together give the extended dynamical definition of nondemolition premeasurement.

Next, the claim of the additional basis-dynamical condition can be put as follows. A given premeasurement is a nondemolition one if and only if, under the assumptions for claim (9a),

\[
\forall k, \forall q_k : U_{AB} \left( |k, q_k \rangle_A |\phi_B^i \rangle \right) \in \mathcal{R}(E_A^k \otimes I_B) \quad (19a)
\]

is satisfied.

**Proof** is obvious (cf that of (9a)).

Conditions (9a) and (19a) can be given the form of one condition:

\[
\forall k, \forall q_k : U_{AB} \left( |k, q_k \rangle_A |\phi_B^i \rangle \right) \in \mathcal{R}(E_A^k \otimes F_B^k) . \quad (19b)
\]

It is the extended basis-dynamical definition of nondemolition premeasurement.

Since \( \dim(\mathcal{R}(E_A^k)) \leq \dim(\mathcal{R}(E_A^k) \otimes \mathcal{R}(F_B^k)) \), the construction of \( U_{AB} \) is always possible. Hence, nondemolition premeasurement of any discrete observable is, in principle, possible (as far as the algebra of unitary premeasurement theory goes, cf concluding remark VI in section X).

The ‘other face of the coin’ of the one relation expressing the basis-dynamical condition (19b), is the extended subspace-dynamical condition of nondemolition premeasurement. It is:

\[
\forall k : U_{AB} \left[ \mathcal{R}(E_A^k) \otimes \mathcal{R}(|\phi_B^i \rangle_B) \right] \subseteq \mathcal{R}(E_A^k \otimes F_B^k) . \quad (19c)
\]

There is another additional system of necessary and sufficient conditions for a premeasurement to be a nondemolition one. It is the twin-observables condition.

A premeasurement is a nondemolition one if and only if its final state satisfies the conditions

\[
\forall |\phi_A^i \rangle, \forall : E_k^A |\Phi_{AB}^k \rangle = F_B^k |\Phi_{AB}^k \rangle \quad (20)
\]

(cf (3), i.e., if and only if all pairs of events \( E_A^k \) and \( F_B^k \) are so-called twin projectors in it, and hence the measured observable \( O_A \) and the pointer observable \( F_B \) are twin observables in it.

**Literature** Twin observables were introduced in Herbst and Vojićić 1976, and elaborated in Herbst 2002. (They are presented in detail in the unpublished review Herbst 2014b.)

**Proof of Necessity** follows from \( \sum_k E_A^k = I_A \) , (3), the calibration condition (6), and the nondemolition condition (16b):

\[
|\Phi_{AB}^k \rangle = \sum_k |E_A^k |\phi_A^i \rangle|_A| |\Phi_{AB}^k \rangle|_B .
\]

To prove sufficiency, let \( |\phi_A^i \rangle = E_A^k |\phi_A^i \rangle \). On account of (6), one has \( |\Phi_{AB}^k \rangle = F_B^k |\Phi_{AB}^k \rangle \). Further, (20) implies \( |\Phi_{AB}^k \rangle = E_A^k |\Phi_{AB}^k \rangle \), i.e., the sharp value of the measured observable is not demolished ((16b) is valid).
One should note that there is no condition for general premeasurement that would be parallel to the 'twin observables condition' (20).

Condition (20) has important consequences. First of all, it implies the additional strong invariance form of the calibration condition (17) in an obvious way (if one has in mind the strong invariance form of the calibration condition (11)). But there is more.

A) In the final subsystem states (reduced density operators)
\[
\rho_A^f \equiv \text{tr}_B\left( |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f \right) \quad \text{and} \quad \rho_B^f \equiv \text{tr}_A\left( |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f \right)
\]
there is no coherence with respect to the events \( E_k^A \) and \( F_k^B \) respectively:
\[
\forall |\psi\rangle_A^1 \in \mathcal{H}_A : \quad \rho_A^f = \sum_k E_k^A \rho_A^f E_k^A, \quad (21a)
\]
\[
\forall |\psi\rangle_A^1 : \quad \rho_B^f = \sum_k F_k^B \rho_B^f F_k^B. \quad (21b)
\]

B) The following commutations are valid:
\[
[E_{A,i}^k, \rho_A^f] = 0, \quad [F_{B,i}^k, \rho_B^f] = 0. \quad (22a,b)
\]

**Proof of claim A):** Since \( \sum_k F_k^B = 1_B \), one has
\[
\rho_A^f = \sum_{k,k'} \text{tr}_B\left( F_{B,i}^k |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f F_{B,i}^{k'} \right).
\]
Utilizing under-the-partial-trace commutativity (1c) twice, and orthogonality of the projectors, one further obtains
\[
\rho_A^f = \sum_{k,k'} \text{tr}_B\left( F_{B,i}^k |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f F_{B,i}^{k'} \right) = \sum_k \text{tr}_B\left( F_{B,i}^k |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f F_{B,i}^k \right).
\]
The twin relations (20) and the fact that opposite-subsystem operators can be taken outside the partial trace (cf (1a,b)) enable one to write further
\[
\rho_A^f = \sum_k \text{tr}_B\left( E_{A,i}^k |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f E_{A,i}^k \right) = \sum_k E_{A,i}^k \text{tr}_B\left( |\Phi\rangle_{AB}^f \langle \Phi|_{AB}^f \right) E_{A,i}^k = \sum_k E_{A,i}^k E_{A,i}^k.
\]
One proves symmetrically, exchanging the roles of the two subsystems and of \( F_k^B \) and \( E_k^A \), that
\[
\rho_B^f = \sum_k F_{B,i}^k F_{B,i}^k.
\]

**Proof of claim B):** It is straightforward to see that claims A and B are equivalent.

One should note that the relation \( \rho_A^f = \sum_{k,k'} E_{k}^A \rho_A^f E_{k'}^A \equiv \mathbb{1}_A \) is an identity on account of the completeness relation \( \sum_k E_k^A = \mathbb{1} \) . Due to the fact that none of the coherence terms \( E_k^A \rho_A^f E_{k'}^A \) \( k \neq k' \) is non-zero (cf (22a)), one says that there is no coherence in the final subsystem state \( \rho_A^f \) as far as the measured observable \( O_A \) is concerned. So it is symmetrically in \( \rho_B^f \). This means that one is dealing with improper mixtures (D’Espagnet, 1976)
\[
\rho_A^f = \sum_k [\text{tr}(\rho_A^f E_k^A)] \left( E_k^A \rho_A^f E_k^A / [\text{tr}(\rho_A^f E_k^A)] \right), \quad \text{etc.}
\]

**Literature**
The twin relations (20) are characterized in Busch and Lahti 1996, sections 4 and 5, as "strong correlations", which are an extremal case of "correlations". The latter are applicable also to general premeasurements. They are described in detail in subsections III.3.1-III.3.4.

Another characterization of nondemolition premeasurement that has no parallel claim for general premeasurements is the claim of repeatability. It asserts that nondemolition complete measurement can be equivalently defined by requiring that, for every initial state of the measured object, immediate repetition of the same complete measurement necessarily (with statistical necessity) gives the same result:
\[
\forall |\psi\rangle_A^1, \forall k, \quad \langle \psi| A E_k^A |\psi\rangle_A^1 > 0:
\]
\[
(\langle \Phi|_{AB}^f F_{B,i}^k E_{A,i}^k A |\Phi\rangle_{AB}^f ) / || F_{B,i}^k |\Phi\rangle_{AB}^f ||^2 = 1. \quad (23a)
\]
Making use of (5), (23a) can be equivalently written as
\[
\forall |\psi\rangle_A^1, \forall k, \quad \langle \psi^1 A | E_k^A |\psi\rangle_A^1 > 0:
\]
\[
E_{A,i}^k F_{B,i}^k |\Phi\rangle_{AB}^f = F_{B,i}^k |\Phi\rangle_{AB}^f. \quad (23b)
\]

**Proof of necessity** If the premeasurement is a nondemolition one, then the twin observables condition (20) is valid. Hence
\[
\forall |\psi\rangle_A^1, \forall k, \quad \langle \psi| A E_k^A |\psi\rangle_A^1 > 0:
\]
\[
E_{A,i}^k F_{B,i}^k |\Phi\rangle_{AB}^f = E_{A,i}^k A |\Phi\rangle_{AB}^f = F_{B,i}^k |\Phi\rangle_{AB}^f, \quad \text{i. e. (23b) is satisfied.}
\]

**Proof of sufficiency** Let \( |\Phi\rangle_A^1 = E_k^A |\Phi\rangle_A^1 \) be valid. Then, utilizing the calibration condition (6) twice, and making use of (23b), we have \( |\Phi\rangle_{AB}^f = E_{A,i}^k F_{B,i}^k |\Phi\rangle_{AB}^f = E_{A,i}^k A |\Phi\rangle_{AB}^f \). Thus, the additional nondemolition condition (16b) for nondemolition premeasurement is satisfied.

**Literature** The 'repeatability condition' is one of the ways how Pauli defined nondemolition measurement in Pauli 1933.
As it was stated at the beginning of this section, also the term "predictive premeasurement" is used as a synonym of "nondemolition premeasurement" in unitary measurement theory of discrete observables. It suggests that the complete measurement result predicts the result of an immediately repeated identical measurement. "Retrodictive" or "retrospective" measurement is then a synonym for "demolition measurement".

There is yet another additional necessary and sufficient condition for nondemolition premeasurement. We shall call it the extended probability reproducibility condition. It reads

$$\forall |\psi\rangle_A, \forall k: \langle \phi|_A E_k^A |\psi\rangle_A = (\Phi^f_{AB} E_k^A |\Phi^f_{AB}\rangle$$  (24)

Proof of necessity follows from the probability reproducibility condition (10):

$$\forall |\psi\rangle_A, \forall k: \langle \phi|_A E_k^A |\psi\rangle_A = (\Phi^f_{AB} F_k^A |\Phi^f_{AB}\rangle$$

Taking into account the validity of the two-observables condition (20), this immediately becomes (24).

To prove sufficiency, we take $|\phi\rangle_A = E_k^A |\phi\rangle_A$. Then (24) implies $(\Phi^f_{AB} E_k^A |\Phi^f_{AB}\rangle = 1$, or, equivalently on account of (5), $E_k^A |\Phi^f_{AB}\rangle = |\Phi^f_{AB}\rangle$. Thus, the sharp value is preserved, and we have nondemolition premeasurement.

**Literature** Pauli (1933) gave the condition at issue the following catchy physical meaning: "The initial state and the final state of premeasurement predict the same probabilities for the measurement in question.

In the book by Busch, Lahti, and Mittelstaedt (1996), which treats a wider class of measurement processes, the 'repeatability condition' and that of the 'extended probability reproducibility condition' are not equivalent: the former is a special case of the latter (see subsections III.3.5 and III.3.6 there). Both in Pauli 1933 and in Busch, Lahti, and Mittelstaedt 1996 the latter measurement is called "of the first kind". And so it is in most of the literature. (The term is abandoned in this review because "first kind" does not suggest the essence of the condition.)

Finally we give two canonical-expansion criteria for nondemolition premeasurement. Let us first specify the subsystem-basis canonical expansion one.

**A** A sufficient additional condition for nondemolition premeasurement reads: A final state $|\Phi^f_{AB}\rangle$ of premeasurement (cf (3)) is that of a nondemolition one if there exists an eigenbasis $\{|k, s_k\}_B : \forall k, s_k\}$ of the pointer observable $P_B$ (cf (2b)) such that, when $|\Phi^f_{AB}\rangle$ is expanded in it, each nonzero expansion coefficient (vector in $\mathcal{H}_A$) is an eigenvector of the measured observable $O_A \equiv \sum_k a_k E_k^A$ with the same $k$-value:

$$\forall |\phi\rangle_A : |\Phi^f_{AB}\rangle = \sum_{k, s_k} |k, s_k\>_A |k, s_k\>_B$$  (25a)

implies

$$\forall k, s_k : E_k^A |k, s_k\>_A = |k, s_k\>_A.$$  (25b)

**B** The final state $|\Phi^f_{AB}\rangle$ is that of nondemolition premeasurement only if (25a,b) is valid for its expansion in every eigen-basis of the pointer observable.

To prove necessity, we point out that (25a) is equivalent to the partial scalar product

$$\forall k : |k, s_k\>_A = \langle k, s_k | B |\Phi^f_{AB}\rangle.$$  

Applying $E_k^A$ to both sides, and utilizing the two-observables condition (20), we obtain

$$\forall k : E_k^A |k, s_k\>_A = \langle k, s_k | B F_k^A |\Phi^f_{AB}\rangle = |k, s_k\>_A.$$  

(Since "partial scalar product" is not used often in the literature, the reader may be unfamiliar with it. Perhaps he (or she) should read Appendix A in Herbut 2011,b.)

To prove sufficiency, we assume the validity of (25a,b), and we apply $E_k^A$ and alternatively $F_k^A$ to (25a) for an arbitrary fixed value $k = \tilde{k}$. Then the two-observables definition (20) follows.

Another criterion for a premeasurement to be a nondemolition one is, what we shall call, the twin-correlated canonical Schmidt decomposition criterion. It is in terms of a special kind of a canonical Schmidt or a bi-orthonormal decomposition with positive numerical expansion coefficients of the final state $|\Phi^f_{AB}\rangle$.

**A** A premeasurement is a nondemolition one if there exists a canonical Schmidt decomposition

$$\forall |\phi\rangle_A : |\Phi^f_{AB}\rangle = \sum_k r_k^{1/2} \sum_{s_k} |k, s_k\>_A |k, s_k\>_B,$$  (26)

where $\{|k, s_k\>_B : \forall k, \forall s_k\}$ are simultaneous eigen-vectors of the pointer observable $P_B \equiv \sum_k p_k F_k^A$, i.e.,

$$\forall k : F_k^B = \sum_{s_k} |k, s_k\>_B \langle k, s_k | B,$$

and the final measuring-instrument state operator (reduced density operator) $\rho_B^f \equiv \text{tr}_A \left( |\Phi^f_{AB}\rangle \langle \Phi^f_{AB}\rangle \right)$ spanning the range $\mathcal{R}(\rho_B^f)$.

Note that symmetrically $\{|k, s_k\>_A : \forall k, \forall s_k\}$ are simultaneous eigen-vectors of the measured observable $O_A \equiv \sum_k a_k E_k^A$ and the final object state operator (reduced density operator) $\rho_A^f \equiv \text{tr}_B \left( |\Phi^f_{AB}\rangle \langle \Phi^f_{AB}\rangle \right)$ spanning the range $\mathcal{R}(\rho_A^f)$.

Actually, (26) is a subsystem-basis expansion with respect to the instrument-subsystem sub-basis $\{|k, s_k\>_A : \forall k, \forall s_k\}$ (spanning $\mathcal{R}(\rho_B^f)$). Hence, the above common eigen-subbasis $\{|k, s_k\>_A : \forall k, \forall s_k\}$ for $O_A$ and $\rho_A^f$ of the object subsystem is uniquely determined by the state $|\Phi^f_{AB}\rangle$.

(The former is determined via the antiunitary so-called "correlation operator" $U_o$ inherent in the composite state. Decomposition (26) is called twin-correlated canonical Schmidt decomposition due to the twin-observables condition.
(20). More about this special kind of Schmidt canonical expansion see in section 5 of Herbut 2014b.)

B) A premeasurement is a nondemolition one only if the twin-correlated canonical Schmidt decomposition (26) is valid for every common eigen-bases of $P_B$ and $P_B'$ that span the range of $\rho_B$.

Proof of sufficiency is immediately obtained by noticing that (26) is a special case of (25a,b). Proof of necessity also follows from this remark and the properties of any twin-correlated canonical Schmidt decomposition. (If in doubt, consult Herbut 2014b.)

**Literature** Twin-correlated canonical Schmidt decompositions were introduced into quantum mechanics by von Neumann (1955, see section 2 in chapter VI, in particular pp. 434-436). Von Neumann did not utilize our term.

The antunitary correlation operator was introduced in Herbut and Vujčič 1976. The canonical Schmidt (or biorthogonal) decomposition of a bipartite state vector (with and without the explicit correlation operator $U$, but without twin correlation) was reviewed in subsection 2.1 of Herbut 2007a.

The twin-correlated canonical Schmidt decomposition criterion for a kind of premeasurement was investigated also by Beltrametti, Cassinelli, and Lahti (1990). They referred to it as to "strong correlations premeasurement".

Some founders of quantum mechanics and many foundationally oriented physicists consider only nondemolition premeasurement for individual quantum systems. The reason is, of course, the fact that unless one can check the result obtained (repeatability), the individual-system physical meaning of the result of premeasurement is doubtful.

Nevertheless nowadays we are witnessing an ever-increasing permeation of physics by information theory (see e. g. Vedral, 2006). General premeasurement, treated in section II, which includes besides nondemolition also demolition premeasurement, transmits information from system to measuring apparatus. In premeasurement this is apparent for individual systems if the initial state has a sharp value of the measured observable; otherwise only probability is transmitted (the probability reproducibility condition), and it is an ensemble phenomenon. But premeasurement underlies, in some way, complete measurement, and the latter transmits individual-system information. Besides, premeasurement theory primarily concerns ensembles. For these reasons thorough investigation of premeasurement more general than the nondemolition one in section II is made the basis of this review.

**V. FUNCTIONS OF THE MEASURED OBSERVABLE; MINIMAL PREMEASUREMENT AND OVERMEASUREMENT**

We denote by $f(\ldots)$ any single-valued real function, i. e., any map of the real axis into itself. It determines a Hermitian operator function $O_A \left(= \sum o_k E_A^k \right)$ of the object subsystem in spectral form as follows:

$$O_A \equiv f(O_A) \equiv \sum_k f(o_k) E_A^k = \sum_l o_l E_A^l,$$

where the second spectral form is, by definition, unique, i. e., it is the same as the first spectral decomposition, but rewritten in the unique form. It satisfies $l \neq l' \Rightarrow o_l \neq o_{l'}$.

In general, the first spectral form is not unique. The inverse function $f^{-1}(\ldots)$ is, in general, multi-valued, i. e., its images are sets: $\forall l : f^{-1}(a_l) = \{o_k : f(o_k) = a_l\}$. We can omit the eigenvalues and keep only their indices because they are in a one-to-one relation (in the unique spectral form). Hence, one can write $\forall l : l = f(k)$, $l$ being the index of $o_l = f(o_k)$, and $\forall l : f^{-1}(l) = \{k : f(k) = l\}$.

Parallelly, one defines a corresponding pointer observable

$$\bar{F}_B \equiv \sum_l p_l F_B^l,$$

where

$$\forall l : F_B^l \equiv \sum_{k \in f^{-1}(l)} F_B^k,$$

and $\{p_l : \forall l\}$ are arbitrary distinct real numbers.

Whenever the function $f(\ldots)$ in (27a-c) is not one-to-one, i. e., whenever this function is singular, the measurement of $O_A$ that is obtained due to a measurement of $O_A$, is called overmeasurement of $O_A$.

If $O_A$ is a complete observable, i. e., if all its eigenvalues are non-degenerate, then $O_A$ is said to be overmeasured maximally (or just "measured maximally" cf Herbut, 1969).

**Literature** In von Neumann 1955 it is mostly assumed that all observables are overmeasured maximally (cf p. 348 not far from the beginning of section 1 of Chapter V).

If an observable is measured so that it is not overmeasured, then one says that it is measured minimally (a term introduced in previous work Herbut, 1969).

**Literature** It was, actually, Lüders (1951) who introduced minimal premeasurement via ideal premeasurement, though he did not call it so. The way I see it, it was an important development in unitary measurement theory after von Neumann. Unfortunately, some physicists still labor under the illusion that if there is any measurement other than maximal, then it is only ideal measurement as Lüders introduced it.

The opposite concept of overmeasurement is undermeasurement. If, in the notation used above, the observable $O_A$ is measured minimally, then $O_A$ (cf (27a)) is undermeasured. The best known example of undermeasurement is that of values in a continuous spectrum. The latter is as, a rule, an interval. One breaks it up into a set-theoretical sum of smaller (non-overlapping) intervals. One evaluates the spectral measures of these subintervals and, utilizing them, one defines a discrete observable. Its measurement undermeasures the continuous spectrum. As it is well known, the values of a continuous spectrum cannot be exactly measured. (There are no eigen-projectors corresponding to them.)

**Literature** Von Neumann has claimed that a continuous spectrum is normally measured undermeasuring it by a discrete observable as just described (cf von Neumann 1955, chapter III, section 3. p. 220). His term for undermeasurement is "measurement with only limited accuracy".
Regarding the continuous spectrum, see also Ozawa (1984).

Now we state and prove the basic claims on overmeasurement:

For every function $f(\ldots)$ the following claims are valid.

A) If $|\phi^i_A\rangle$ is calibration-condition-satisfying for the premeasurement of $O_A$, then so is also for that of $O_A \equiv f(O_A)$ in terms of the pointer observable $P_B$ (cf (27a-c)). Remembering the dynamical definition of general measurement, this claim can read:

$$\forall \phi^i_A, \forall k: \quad F^k_B U_B A\left(|\phi^i_A\rangle|\phi^i_B\rangle\right) = U_B E^k_A \left(|\phi^i_A\rangle|\phi^i_B\rangle\right) \Rightarrow$$

$$\forall l \left( \equiv f(k) \right): \quad F^l_B U_B A\left(|\phi^i_A\rangle|\phi^i_B\rangle\right) = U_B E^l_A \left(|\phi^i_A\rangle|\phi^i_B\rangle\right).$$

B) If the premeasurement of $O_A$ is a nondemolition one, then so is that of its function $O_A \equiv f(O_A)$ . Having in mind the definition of nondemolition premeasurement in terms of twin observables (20), this additional claim can be put as follows:

$$\forall \phi^i_A, \forall k: \quad F^k_B |\Phi^i_{AB}\rangle = E^k_A |\Phi^i_{AB}\rangle \Rightarrow$$

$$\forall l \left( \equiv f(k) \right): \quad F^l_B |\Phi^i_{AB}\rangle = E^l_A |\Phi^i_{AB}\rangle. \quad \text{(28a)}$$

To prove claim A, we argue as follows. On account of the assumption that the calibration condition (6) is valid for the premeasurement of $O_A$, one has

$$\forall \phi^i_A: \quad |\Phi^i_{AB}\rangle = U_B A\left(|\phi^i_A\rangle|\phi^i_B\rangle\right) = \sum_k ||E^k_A \phi^i_A|| \times F^k_B U_B A\left(E^k_A \phi^i_A |E^k_A \phi^i_B\rangle\right).$$

Hence,

$$\forall \phi^i_A: \quad |\Phi^i_{AB}\rangle = \sum_k F^k_B U_B A\left(E^k_A \phi^i_A |\phi^i_B\rangle\right). \quad \text{(aux)}$$

Further, we rewrite this as two sums, and we take $|\phi^i_A\rangle = E^i_A \phi^i_A$. Then we replace $|\phi^i_A\rangle$ by $E^i_A \phi^i_A$ in the second sum, obtaining

$$|\Phi^i_{AB}\rangle = \sum_{k \in f^{-1}(l)} F^k_B U_B A\left(E^k_A \phi^i_A |\phi^i_B\rangle\right) +$$

$$\sum_{k \in f^{-1}(l)} E^k_A F^k_B U_B A\left(E^k_A \phi^i_A |\phi^i_B\rangle\right).$$

Next, we take into account the facts that $E^i_A E^k_A = E^k_A$ if $k \in f^{-1}(l)$, $E^i_A E^k_A = 0$ if $k \notin f^{-1}(l)$ and correspondingly (cf (27a-c)) $F^k_B F^k_B = F^k_B$ if $k \in f^{-1}(l)$, $F^k_B F^k_B = 0$ if $k \notin f^{-1}(l)$ . We thus obtain

$$|\Phi^i_{AB}\rangle = F^i_B \sum_{k \in f^{-1}(l)} F^k_B U_B A\left(E^k_A \phi^i_A |\phi^i_B\rangle\right) =$$

$$F^i_B \sum_k F^k_B U_B A\left(E^k_A \phi^i_A |\phi^i_B\rangle\right).$$

(We were able to include the terms for $k \notin f^{-1}(l)$ because they are zero.) Finally, on account of the above general auxiliary relation (aux), we have in our special case

$$|\Phi^i_{AB}\rangle = F^i_B |\Phi^i_{AB}\rangle.$$

Therefore, the calibration condition for $O_A$ is valid (cf (6)), and we are dealing with a premeasurement of this observable.

Proof of claim B We assume that we have nondemolition premeasurement of $O_A$, and we utilize the twin-relations definition (20) for this. Then, on account of (27a-c),

$$\forall l: \quad F^l_B |\Phi^i_{AB}\rangle = \left( \sum_{k \in f^{-1}(l)} F^k_B \right) |\Phi^i_{AB}\rangle =$$

$$\left( \sum_{k \in f^{-1}(l)} E^k_A \right) |\Phi^i_{AB}\rangle = E^l_A |\Phi^i_{AB}\rangle$$

follows for an arbitrary initial state of the object. We see that the premeasurement is nondemolition also for that of $O_A$ because the twin relations (20) are valid also for this observable.

VI. DISENTANGLED PREMEASUREMENTS

Now the question of inverting the described functional relation $O_A \equiv f(O_A)$ (cf (27a-c)) arises. For any answer there is a need for a more precise terminology. We have seen that the eigenvalues do not play an essential role in premeasurement; only the eigen-projectors and their indices do.

Having in mind the relation between the indices $k$ and $l$ (cf (27a-c)), one says, by definition, that $O_A$ is coarser than $O_A$, or that the former is a coarsening of the latter.

The same relation can be expressed also by the terms: $O_A$ is finer than $O_A$, or the former is a refinement of the latter.

If a coarsening $O_A \equiv f(O_A)$ (cf (27a-c)) is mathematically given, and one measures the coarser observable $O_A$ minimally, one may wonder if this means anything for the finer observable $O_A$. Clearly, the answer is: one has an undermeasurement of $O_A$. This would belong to approximate measurement theory, which is outside the scope of this article.

It may happen that a given premeasurement of an observable $O_A$ is actually a premeasurement of one of its refinements, i.e., that $O_A = f(O_A)$ is valid, and that $O_A$ is overmeasured. If this is the case, then, as it was explained, there must exist also a corresponding refinement of the pointer observable $P_B$ with co-indexed eigen-projectors, and a subspace of the subspace spanned by all the calibration-condition-satisfying states $|\phi^i_B\rangle$, the unit vectors in which are calibration-condition-satisfying for the refinement. Evidently, it is not always easy to find out if a given observable $O_A$ is actually overmeasured.

We now turn to a class of cases in which, though $O_A$ itself may have refinements, its given measurement is certainly not an overmeasurement, i.e., it is a minimal measurement.

By definition, one has a disentangled or uncorrelated premeasurement of an observable $O_A$ if there exists an eigen-sub-basis $\{|\phi^i_B\rangle, \forall k, |\phi^i_B\rangle = F^i_B |\phi^i_B\rangle\}$
of the pointer observable $P_B$ (cf (2b)), and for each $k$ value only one eigenvector $|\phi^k_B\rangle$ from this subbasis appears in the canonical subsystem-basis expansion (13a) of the final premeasurement state. Equivalently, for every initial state $|\phi^i_A\rangle$ of the object, the final state $|\Phi^i_{AB}\rangle = U_{AB}\left(|\phi^i_A\rangle \otimes |\phi^k_B\rangle\right)$ can be expanded in this sub-basis:

$$\forall \ |\phi^i_A\rangle : \ |\Phi^i_{AB}\rangle = \sum_k |\phi^k_A\rangle \otimes |\phi^i_B\rangle,$$

(29a)

where $|\phi^k_A\rangle$ are vectors in $\mathcal{H}_A$. Then, equivalently

$$\forall k : \ |\Phi^k_{AB}\rangle = \langle \phi^i_B | |\Phi^i_{AB}\rangle,$$

(29b)

the right-hand sides being orthonormal scalar products (cf possibly Appendix A in Herbut 2014b).

The term ‘disentangled’ (introduced by Schrödinger, 1935) applies to the uncorrelated (disentangled) vectors $|\phi^k_A\rangle \otimes |\phi^i_B\rangle$ in the final state (29a).

Disentangled premeasurement can be made from the nondecomposition kind, when the result of measurement is preserved throughout the branches $\forall k : |\phi^k_A\rangle = E^k_A |\phi^k_A\rangle$, or of the decomposition kind if the preservation fails for some $k$ value. Among the latter, there is an interesting possibility: the set of ‘expansion coefficients’ $\{\forall k : |\phi^k_A\rangle\}$ in (28a) may turn out orthogonal $\forall |\phi^i_A\rangle$. Then they determine another observable of subsystem $A$ with sharp values in the selective final states $F^k_B |\Phi^i_{AB}\rangle / ||F^k_B |\Phi^i_{AB}\rangle||$ possibly independently of the initial state.

It is worth mentioning that, whereas most measurement properties are valid (or not valid) for each branch separately; the property at issue is a premeasurement property, valid for the entirety of the final state.

**Literature** This case is referred to as “strong state correlation” in Busch, Lahti, and Mittelstaedt 1996 (subsection III.3.2).

A sufficient condition for disentangled premeasurement appears in nondecomposition premeasurement of a complete observable $O_A : \mathcal{O}_A = \sum_k |k\rangle_A \langle k|_A$, $k \neq k' \Rightarrow \langle k|_A \neq \langle k'|_A$. Then the preservation of $E^k_A = \langle k|_A \langle k|_A$ requirement implies that in each complete-measurement branch $F^k_B |\Phi^i_{AB}\rangle$ only $|k\rangle_A$ can appear (cf (20)): $|\Phi^k_{AB}\rangle = \sum_k |k\rangle_A \otimes |\phi^k_B\rangle$. (The state vector $|k\rangle_A$, as any state vector, cannot be entangled.)

In many discussions of the paradox of the emergence of complete measurement in quantum mechanics, one confines oneself to the simplified case of disentangled measurement. (Most likely, one keeps unitary dynamics as simple as possible in order not to cloud the issue of the paradox.)

Another sufficient condition for disentangled premeasurement is non-degeneracy of all ‘pointer positions’ for obvious reasons (cf in section 2 the passage next to the one in which relation (3) is).

**Literature** In Busch, Lahti, and Mittelstaedt 1996 this case is referred to as “the pointer observable is minimal in the sense that it is just sufficient to distinguish between the eigenvalues” of the measured observable (in passage next to (7) in III.2.3).

Let us next state and prove the basic property of disentangled measurements. It goes as follows.

If a given premeasurement of an observable $O_A$ is disentangled, then the evolution operator $U_{AB}$ can be replaced by a set of partial isomorphisms $\{U^k_B : \forall k\}$ each mapping the range $\mathcal{R}(E^k_A)$ of the corresponding eigen-projector $E^k_A$ (cf (1a)) into $\mathcal{H}_A$ isometrically, i.e., linear and preserving scalar products. The replacement is performed so that

$$\forall |\phi^i_A\rangle : |\Phi^i_{AB}\rangle \equiv U_{AB}(|\phi^i_A\rangle \otimes |\phi^k_B\rangle) = \sum_k (U^k_B E^k_A |\phi^i_A\rangle \otimes |\phi^k_B\rangle)$$

(30)

is valid.

One should note that the operators $\{U^k_B : \forall k\}$, being partial isometries can be extended into a unitary operator in $\mathcal{H}_A$, and they map orthonormal vectors into orthonormal ones.

To prove the claim, let us take a complete orthonormal eigen-basis $\{|k,q_A\rangle : \forall k,q\}$ of $O_A$, and expand $|\phi^i_A\rangle = \sum_{k,q} \langle k,q |\phi^i_A\rangle |k,q\rangle_A \otimes |k,q\rangle_B$. Next, we evaluate the final premeasurement state using this expansion.

$$U_{AB}(|\phi^i_A\rangle \otimes |\phi^k_B\rangle) = \sum_{k,q} \langle k,q |\phi^i_A\rangle |k,q\rangle_A \times U_{AB}(|k,q\rangle_A \otimes |\phi^k_B\rangle).$$

According to the calibration condition (6), each final state $U_{AB}(|k,q\rangle_A \otimes |\phi^k_B\rangle)$ must be invariant under the action of $F^k_B$ on the one hand, and, according to the definition of disentangled premeasurement, only one of the given eigen-vectors $|\phi^k_B\rangle$ independent of the $q$ values, can appear in the expansion. Hence, $U_{AB}$, being unitary, maps, for each value of $k$, the orthonormal sub-basis $\{|k,q\rangle_A \otimes |\phi^k_B\rangle : \forall q\}$ into some other orthonormal sub-basis $\{|k,q\rangle_A \otimes |\phi^k_B\rangle : \forall q\}$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. (Note that the state vectors $|\phi^k_B\rangle$, as any state vectors, cannot be entangled.) It follows from the orthonormality of $\{|k,q\rangle_A \otimes |\phi^k_B\rangle : \forall q\}$ that also the set $\{|k,q\rangle_A \otimes |\phi^k_B\rangle : \forall q\}$ is orthonormal.

Each pair of orthonormal sub-bases determines a partial isometry $\forall q, U^k_A : \forall q : U^k_A |k,q\rangle_A = |k,q\rangle_A$. Hence, we can write

$$U_{AB}(|\phi^i_A\rangle \otimes |\phi^k_B\rangle) = \sum_{k,q} \langle k,q |\phi^i_A\rangle \times (U^k_A |k,q\rangle_A) |\phi^k_B\rangle = \sum_k (U^k_B E^k_A |\phi^i_A\rangle \otimes |\phi^k_B\rangle).$$

The characterization of disentangled premeasurement by (30) is the essence of this kind of premeasurement. The operators $\{U^k_B E^k_A : \forall k\}$ are called state transformers because, in disentangled complete measurement with the result $\alpha_k$, they transform the initial state $|\phi^i_A\rangle$ into
As it is easily seen, any disentangled premeasurement is a non-demolition one if and only if
\[ \forall k : \quad U_k^A E_A^k | \phi_A^k \rangle = E_A^k U_k^A | \phi_A^k \rangle; \] (31)
otherwise, it is a demolition premeasurement (cf (16b)).

VII. IDEAL PREMEASUREMENTS

The simplest among disentangled premeasurements are the ideal premeasurements. They can be defined in several equivalent ways. The ones that are most used are the following.

(I) \[ \forall k : \quad U_k^A = I_k \] , where \( I_k \) is the identity operator in \( \mathcal{H}_A \) restricted to the range \( \mathcal{R}(E_A^k) \) of \( E_A^k \) (cf (30)): \[ \forall | \phi_A^k \rangle : \quad | \phi_A^k \rangle = \sum_k (E_A^k | \phi_A^k \rangle) \otimes | \phi_B^k \rangle. \] (32)

Expansion (32) is obviously a twin-correlated canonical Schmidt (or bi-orthogonal) decomposition (having the measured observable and the pointer observable as twin observables in mind, cf (20) and (26)). We may call it shortly the canonical-final-state definition of ideal premeasurement.

(II) The definition that ensues may be called the Lüders change-of-state one. It says that the general final state \( \rho_A^f \) of the object in ideal premeasurement of an observable \( O_A \) \[ = \sum_k o_k E_A^k \] is given by:

\[ \forall | \phi_A^k \rangle : \quad \rho_A^f = \text{tr}_B \left( | \Phi_{AB}^f \rangle \langle \Phi_{AB}^f | \right) = \sum_k E_A^k | \phi_A^k \rangle | \phi_A^k \rangle \langle \phi_A^k | E_A^k. \] (33a)

One can rewrite (33a) as

\[ \forall | \phi_A^k \rangle : \quad \rho_A^f = \text{tr}_B \left( | \Phi_{AB}^f \rangle \langle \Phi_{AB}^f | \right) = \sum_k (E_A^k | \phi_A^k \rangle | \phi_A^k \rangle \langle \phi_A^k | E_A^k) / \text{tr}(E_A^k | \phi_A^k \rangle | \phi_A^k \rangle \langle \phi_A^k | E_A^k), \] (33b)
as seen if one applies, in the denominator, under-the-trace commutativity, idempotency, and if one evaluates the trace in a basis containing \( | \phi_A^k \rangle \).

In this way it is seen that the final complete-measurement states are the pure states

\[ \forall k, \quad (E_A^k | \phi_A^k \rangle | \phi_A^k \rangle > 0 : \quad E_A^k | \phi_A^k \rangle / ||E_A^k | \phi_A^k \rangle||, \] (33c)
where \( (E_A^k | \phi_A^k \rangle | \phi_A^k \rangle \) are, of course, the probabilities in the improper mixture (33b) (cf D’Espagnat 1976).

(III) The third definition may be called the strongly extended calibration condition. It reads: Every initial state that has a sharp value of the measured observable does not change at all in ideal premeasurement:

\[ | \phi_A^k \rangle = E_A^k | \phi_A^k \rangle \Rightarrow \rho_A^f = (E_A^k | \phi_A^k \rangle \langle \phi_A^k |. \] (34a)

This definition of ideal premeasurement is in line with our first definitions of general premeasurement (6) and non-demolition premeasurement (16b). Comparing (34a) with (16b), it is obvious that ideal premeasurement is a special case of non-demolition premeasurement.

One should note that the first definition of ideal premeasurement is not additional to that of general premeasurement; it defines the premeasurement in question by itself completely. The second and third definitions, on the contrary, are additional requirements because premeasurement cannot be defined only by changes in the object subsystem.

We give an ‘in-circle’ proof (to be distinguished from a circular one) of the equivalence of the three definitions.

The first implies the second: Substituting the final premeasurement state from (32) both in its ket and bra forms, one obtains

\[ \rho_A^f = \text{tr}_B \left( | \Phi_{AB}^f \rangle \langle \Phi_{AB}^f | \right) = \sum_k E_A^k | \phi_A^k \rangle | \phi_A^k \rangle \langle \phi_A^k | E_A^k = \text{RHS}(33a). \]

The second implies the third: Substituting \( | \phi_A^k \rangle = E_A^k | \phi_A^k \rangle \) in (33a) immediately lads to \( \rho_A^f = (E_A^k | \phi_A^k \rangle \langle \phi_A^k |. \)

The third implies the first. The third definition (34a) can be completed into

\[ | \phi_A^k \rangle = E_A^k | \phi_A^k \rangle \Rightarrow | \phi_A^f \rangle = (E_A^k | \phi_A^k \rangle \langle \phi_A^k | \] (34b)

with

\[ | \phi_B^k \rangle = F_B^k | \phi_B^k \rangle. \]

Relations (34b,c) imply that each complete-measurement branch \( E_A^k | \phi_A^k \rangle / ||E_A^k | \phi_A^k \rangle|| \) in the decomposition \( | \phi_A^k \rangle = \sum_k E_A^k | \phi_A^k \rangle \) gives a disentangled component in the final state. Hence, the final premeasurement state is disentangled. Having in mind \( I_A = \sum_k E_A^k \) , and (34b) with (34c), one obtains:

\[ | \phi_A^f \rangle = \sum_k U_{AB} \left( E_A^k \left| \phi_A^k \right\rangle \left| \phi_B^k \right\rangle \right) = \sum_k \left| E_A^k | \phi_A^k \rangle \right| \times U_{AB} \left( E_A^k \left| \phi_A^k \right\rangle \left| \phi_B^k \right\rangle \right) = \sum_k \left( E_A^k \left| \phi_A^k \right\rangle \left| \phi_B^k \right\rangle \right) = \sum_k \left( E_A^k \left| \phi_A^k \right\rangle \right) \left| \phi_B^k \right\rangle. \]

(34c)
L"uders change of state coincides with that postulated by von Neumann (1925) as his process 1. Therefore ideal measurement is sometimes called von Neumann-L"uders measurement.

Among others, also the present author studied some derivations of the L"uders change of state Herbut (1969), Herbut (1974), Herbut (2007b). Also Khrennikov (2009a, 2008, and 2009b) has paid great attention to ideal measurement.

Incidentally, the postulate of L"uders (33a-c) was called ‘minimal measurement’ in Herbut, 1969 to stress the fact that it has introduced a difference with respect to von Neumann (1955), who restricted his discussion to maximal overmeasurement of degenerate observables, i.e., measurement of complete observables whose function the initial observable is.

Ideal measurement is not very important in practice because it cannot be achieved in direct measurement. The characteristic property (34a) contradicts the empirical fact that in every direct interaction at least one quantum of action has to be exchanged, i.e., some change in the state has to be brought about. But this measurement is of great theoretical importance. This will be elaborated elsewhere.

VIII. SUBSYSTEM MEASUREMENT AND DISTANT MEASUREMENT

In this section the main results of a recent article (Herbut 2014c) are shortly reviewed to complete the picture on basic measurement state

\[
|\Phi_{A_1,A_2,B}^{f}\rangle = U_{A_1} U_{A_2,B} \left( |\phi_{A_1,A_2}^{i}\rangle \otimes |\phi_{B}^{i}\rangle \right)
\]

\[
U_{A_1} \text{ being the unitary evolution operator of the dynamically isolated subsystem } A_1 \text{ that is ‘untouched’ by the measurement interaction.}
\]

Actually, even more is true. It is a known fact that any unitary change to the nearby subsystem \(A_2\), with or without an ancilla \(A_3\), does not have any influence on the state of a dynamically isolated subsystem \(A_1\). (For details, see section 4 in Herbut 2014c.)

If, on the other hand, one considers the final state

\[
F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \left/ \left| |F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle\right| \right.
\]

of a complete measurement, then there is, in general, a change in the state of the ‘interactionally untouched’ distant subsystem \(A_1\), which is due to the entanglement in the initial state \(|\phi_{A_1,A_2}^{i}\rangle\).

Then, the claim is that the final distant-subsystem state in question has the form:

\[
\rho_{A_1}^{f,k} = U_{A_1} \left( \rho_{A_1} (E_{A_2}^{k}) U_{A_1}^{\dagger} \right)
\]

where

\[
\rho_{A_1} (E_{A_2}^{k}) \equiv \text{tr}_{A_2} \left( |\phi_{A_1,A_2}^{i}\rangle \langle \phi_{A_1,A_2}^{i}| E_{A_2}^{k} \right) / \| E_{A_2}^{k} \| \langle \phi_{A_1,A_2}^{i} | E_{A_2}^{k} \|
\]

is denoted the conditional state of the distant subsystem \(A_1\) under the condition of the ideal occurrence of the event \(E_{A_2}^{k}\) in the composite-system state \(|\phi_{A_1,A_2}^{i}\rangle\) instantaneously at the initial moment. This interpretation of (36b) is obvious if one rewrites the relation (utilizing idempotency and under-the-partial-trace commutativity (1c)) in the equivalent form

\[
\rho_{A_1} (E_{A_2}^{k}) = \text{tr}_{A_2} \left( E_{A_2}^{k} |\phi_{A_1,A_2}^{i}\rangle \langle \phi_{A_1,A_2}^{i}| E_{A_2}^{k} \right) / \| E_{A_2}^{k} \| \langle \phi_{A_1,A_2}^{i} | E_{A_2}^{k} \|
\]

utilizing the trivial equality

\[
\text{tr} \left( |\phi_{A_1,A_2}^{i}\rangle \langle \phi_{A_1,A_2}^{i}| E_{A_2}^{k} \right) = \| E_{A_2}^{k} \| \langle \phi_{A_1,A_2}^{i} | E_{A_2}^{k} \|
\]

and having in mind relation (33c) for ideal measurement. (The lengthy proof of claim (36a,b) is given in the Appendix of Herbut 2014c.)

On account of the fact that the unitary interaction operator \(U_{A_2,B}\) has dropped out of the final expression (36a,b), it is an important corollary of the result that, whatever the kind of complete measurement performed on \(A_2\), the change caused to \(A_1\) is the same as if the subsystem measurement were ideal.

The change of state due to ideal complete measurement is easily evaluated on account of relation (32) and the probability reproducibility condition \(\| F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \|^2 = \| E_{A_2}^{k} \| \langle \phi_{A_1,A_2}^{i} | E_{A_2}^{k} \|^2\) (cf (10)):

\[
F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle = \| F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \left/ \left| \| F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \right| \right.
\]

As to the state of the distant subsystem \(A_1\), one has:

\[
\rho_{A_1}^{f,k} \equiv \text{tr}_{A_2,B} \left[ (F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \left/ \left| \| F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \right| \right. \right] \left/ \left( \| F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \left/ \left| \| F_{B}^{k} |\Phi_{A_1,A_2,B}^{f}\rangle \right| \right. \right) \right]
\]

Further, substituting here (38) and taking into account (37), one finally obtains

\[
\rho_{A_1}^{f,k} = U_{A_1} \left[ \text{tr}_{A_2,B} \left( E_{A_2}^{k} |\phi_{A_1,A_2}^{i}\rangle \langle \phi_{A_1,A_2}^{i}| E_{A_2}^{k} \right) \right] \left/ \left( \| E_{A_2}^{k} \| \langle \phi_{A_1,A_2}^{i} | E_{A_2}^{k} \|^2 \right) \right. \right] \left/ \left( \| E_{A_2}^{k} \| \langle \phi_{A_1,A_2}^{i} | E_{A_2}^{k} \|^2 \right) \right. \right]
\]
Now we come to **distant measurement**. We assume that initially we have a twin-observables relation
\[ \forall k : \ E_{A_1}^k \ |\phi\rangle_{1A_1} = E_{A_2}^k \ |\phi\rangle_{1A_2}, \]  
where \( E_{A_i}^k \) are the eigen-projectors of a distant-subsystem observable \( O_{A_i} \) \( = \sum_k \bar{o}_k E_{A_i}^k \). (The twin-observables criterion for nondemolition observables (20) is an example for (40)).

Making use of (40), relation (39) implies
\[ \rho_{A_1}^k = U_{A_1} \left[ \text{tr}_{A_2} \left( E_{A_1}^k \ |\phi\rangle_{1A_1} \langle\phi|_{1A_1} E_{A_1}^k \right) \right] / \ ||E_{A_1}^k \ |\phi\rangle_{1A_1}||^2 U_{A_1}^{-1}. \]  
in view of (33c) for ideal measurement, relation (41) gives the final distant-subsystem state due to the instantaneous ideal occurrence of the (twin) distant subsystem event \( E_{A_1}^k \) in the state \( |\phi\rangle_{1A_2} \) at the initial moment.

Comparing (39) and (41), one can see that both the instantaneous ideal occurrence of the event \( E_{A_1}^k \) on the nearby subsystem and that of the event \( E_{A_2}^k \) on the distant subsystem lead to the same final state \( \rho_{A_1}^k \) of the distant subsystem \( A_1 \). In other words, one can say that the instantaneous ideal occurrence of \( E_{A_2}^k \) in the state \( |\phi\rangle_{1A_1} \) at the initial moment gives rise to the instantaneous ideal occurrence of the distant twin event \( E_{A_1}^k \) in the same state at the same moment.

This was introduced and called distant measurement a long time ago (Herbut and Vujićić 1976). But now we have the recent (above mentioned) result that any exact nearby-subsystem complete measurement leads to the same final state of the distant subsystem as nearby-subsystem complete measurement. Hence, one can say that, if twin observables are involved (cf (40)), any exact subsystem measurement gives rise to a change of state in the opposite subsystem which is the same as caused by instantaneous ideal measurement of the corresponding distant twin event at the initial moment in the given composite-system state \( |\phi\rangle_{1A_2} \) with entanglement. We can keep the term "distant measurement" for this more general measurement.

One should note that as long as the nearby-subsystem complete measurement is ideal, it gives the same change of the global (composite) state as the analogous ideal complete measurement on the distant subsystem. If the former complete measurement is more general than ideal, then its effect coincides with that of ideal complete measurement on the distant subsystem only locally on the distant subsystem.

Distant measurement plays a natural role in the paradoxical so-called Einstein-Podolsky-Rosen (EPR) phenomenon. If a bipartite state vector \( |\phi\rangle_{1A_2} \) allows distant measurement of two mutually incompatible observables (non-commuting operators) \( O_{A_2} \) and \( \bar{O}_{A_2} \), \( [O_{A_2}, \bar{O}_{A_2}] \neq 0 \), then we say that we are dealing with an EPR state (following the seminal Einstein-Podolsky-Rosen 1935 article).

A very simple example of an EPR state is the well known singlet two-particle spin state
\[ |\phi\rangle_{1A_2} = (1/2)^{1/2} \left( (|+\rangle_{A_1} |-\rangle_{A_2} - |-\rangle_{A_1} |+\rangle_{A_2} \right), \]  
where + and − denote spin-up and spin-down respectively along any fixed axis. One can see, in obvious notation, that the oppositely oriented spin-projection operators are twin observables:
\[ g_{A_0A_1} |\phi\rangle_{1A_2} = g_{(-A_0),A_2} |\phi\rangle_{1A_2}, \]  
where the unit vector \( \vec{k}_0 \) is arbitrary. One can suitably choose \( \vec{k}_0 \) either along the positive z-axis or along the positive x-axis.

If one performs an ideal complete measurement of the spin projection along the z-axis in a subsystem measurement on the nearby subsystem \( A_2 \), it is easily seen that the composite final state is, e.g., \( |z,+\rangle_{A_1} |z,−\rangle_{A_2} \), and the (this time pure) state of the distant subsystem \( A_1 \) is \( |z,+\rangle_{A_1} \). Analogously, one can obtain by distant complete measurement, e.g., \( |x,+\rangle_{A_1} \).

Einstein et al. pointed out that in transition from (42) to the final state of distant complete measurement, the mentioned result of opposite spin projection along the same axis was brought about in a distant action without interaction (a "spooky" action), which could not be reconciled with basic physical ideas that reigned outside quantum mechanics. (More in section 6.2 of Herbit 2014b or in Bohm 1952, chapter 22, section 15. and further.)

One can find articles in the literature in which all entangled bipartite states are called EPR states. Perhaps because any entanglement allows distant complete measurement (with intuitively "spooky" action).

**VIII. CLASSIFICATION**

In the classification that follows we disregard overmeasurements because in them the measured observable is a function of a finer observable, and the measurement is just a consequence of the minimal measurement of the latter.

Five kinds of final complete-measurement components of minimal measurements can be distinguished, and they are displayed in the ARRAY below, and denoted by \( M_{x,y} \). They are (in reading order on the Diagram):

1) the **ideal ones**, \( M_{1,1,0} \), characterized by preserving the sharp-value component states (cf (34a));
2) the **nondemolition non-ideal entangled ones**, \( M_{1,1,1} \), which do not preserve the sharp-value component states, but they preserve the sharp values themselves (cf (16b)), and they map the initial pure states \( |\phi\rangle_A \) into pure states
\[ U_A^k \left( E_{A}^k \ |\phi\rangle_A \right)/ ||E_{A}^k \ |\phi\rangle_A|| = I_A; \]  

3) the **nondemolition entangled ones**, \( M_{1,2} \), which also preserve the mentioned sharp values, but they map the initial pure states into mixtures (improper or second-kind ones cf D’Espagnat, 1976)
\[ \rho_{A}^k \equiv \text{tr}_B \left[ U_{AB}(|\phi\rangle_A \langle\phi|_B)(\langle\phi|_A \langle\phi|_B)U_{AB}^\dagger \right]. \]  

In other words, redundant entanglement is created between the subsystems \( A \) and \( B \) (‘redundant’ regarding the premeasurement).
4) the demolition disentangled ones, $M_{2,1}$, which do not preserve the sharp values, but they do map the initial states $|\phi_i^0\rangle_A$ into pure states given by (44); and finally

5) the demolition entangled ones, $M_{2,2}$, which neither preserve the sharp values, nor do they map the pure states into pure states. They map the former into improper mixtures given by (45), i.e., they create redundant entanglement.

<table>
<thead>
<tr>
<th>ARRAY (with rows and columns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DISENTANGLED:</td>
</tr>
<tr>
<td>NONDEMOLITION:</td>
</tr>
<tr>
<td>DEMOLITION:</td>
</tr>
</tbody>
</table>

To understand the ARRAY, one must take into account that a kind of measurement $M_{x,y}$ is defined by the row $x$ and the column $y$. In the special case of $M_{1,1}$, one has two kinds: $M_{1,1,a}$, which is ideal measurement, and $M_{1,1,b}$, which is non-ideal.

One can utilize the classification into $M_{x,y}$ complete measurements also for premeasurements if they are homogeneous in the sense that all the final complete-measurement components of the premeasurement belong to one and the same kind $M_{x,y}$ of the 5 complete measurements. Those that lack such homogeneity can be classified by the worst final complete-measurement component, "worst" meaning that it is farthest from the beginning in reading order on the array.

**IX. SUMMING UP THE EQUIVALENT DEFINITIONS**

Let us sum up that we have obtained 7 equivalent definitions of general premeasurement in section 2:

1) the 'statistical form' of the calibration condition (4),
2) its 'invariance form' (6),
3) its 'strong invariance' form (11);
4) the 'probability reproducibility condition' (10),
5) the 'basic dynamical' characterization (8),
6) the 'basis-dynamical' characterization (9a), and, as the other side of the coin, the 'subspace-dynamical' criterion (9b), and finally
7) the canonical subsystem-basis expansion criterion (34a) with (13b).

To the author's knowledge, new are 3), 5) and 7).

Let us sum up the results of section 4. We have defined non-demolition premeasurement by additional requirements in 10 equivalent ways. We have extended the 7 equivalent definitions of general premeasurement, and thus we have obtained:

1) the extended statistical calibration condition definition (4) + (16a),
2) the extended invariance calibration condition criterion (6) + (16b),
3) the extended strong invariance characterization (11) + (17),
4) the extended probability reproducibility condition (10) + (24),
5) the extended dynamical' definition (8) + (18),

6) the extended basis-dynamical' characterization (9)+ (19a) or equivalently (19b) by itself; further, the extended subspace-dynamical' condition definition (19c);
7) the canonical subsystem-basis expansion criterion (25a) with (25b), and
8) the free-corrected canonical Schmidt decomposition' definition (26).

The canonical subsystem-basis expansion' characterization of general premeasurement (13a) with (13b) was extended in two ways (items 7) and 8).

Two additional conditions without a counterpart in general premeasurement were given:
9) The Pauli 'definition of repeatability' (23a) or (23b), and
10) the 'twin-observables relation' (20).

As far as the author can tell, new are 3) and 8).

Characterization in item 10) had the consequences of lack of coherence (21a) in the final object state with respect to the measured observable $O_A$ and analogously (21b) regarding the final measuring-instrument state and the pointer observable $P_B$.

Finally, let us sum up the three equivalent definitions of ideal premeasurement:

1) The canonical-final-state definition (32),
2) the L"{u}ders-change-of-state definition (33a-c), and
3) the strongly extended calibration condition definition (34a-c).

**X. CONCLUDING REMARKS**

Let us conclude this review by a few remarks on some aspects that have been omitted. What has been covered is outlined at the very beginning of the article.

(I) Complete measurement was viewed only as a constituent of premeasurement.

**Literature** A quote from the first passage of the last chapter of the book on quantum mechanics by Peres (2002) reads:

"In order to observe a physical system, we make it interact with an apparatus. The latter must be described by quantum mechanics, because there is no consistent dynamical scheme in which a quantum system interacts with a classical one. On the other hand, the result of the observation is recorded by the apparatus in a classical form ..."

Transition from quantum-mechanical description of the measuring instrument to that of classical physics Peres calls "dequantization". It seems to be way out of the scope of unitary quantum measurement theory. See also Hay and Peres, 1998.

Among other things, the expounded theory allows generalization (extension) in a number of its basic concepts that have not been covered.

(II) Both the initial state of the object $|\phi_i^0\rangle_A$ and that of the measuring instrument $|\psi_j^0\rangle_B$ allow generalization to general states (density operators) $\rho_A$ and $\rho_B$ respectively.

It is to be expected that, as it is usually the case with density operators, the relations valid for general states can easily be obtained from the corresponding relations valid for pure states when one decomposes the general states into pure...
ones. Besides, there is so-called purification: Every density operator can be viewed as the reduced density operator of a bipartite pure state.

**Literature** As to general states, von Neumann has proved in his famous no-go theorem (von Neumann, 1955, first part of section 3. in chapter VI) that unitary measurement theory cannot explain complete measurement, i.e., the fact that the individual systems end up in one branch

$$F_B^f | \Phi \rangle_{AB} / ||F_B^f | \Phi \rangle_{AB}||$$  \hspace{1cm} (46)

of the final premeasurement state $| \Phi \rangle_{AB}$ (if the complete measurement is a minimal one, cf. section V). We outline the claim of this theorem.

In a purely pure-state measurement theory, as in the present review, it is clear that the coherence $| \phi \rangle_A = \sum_k E_A^k | \phi \rangle_A^k$ with respect to the eigenvalues $\alpha_k$ of the measurable observable $O_A \left( = \sum_k \alpha_k E_A^k \right)$ in the initial state $| \phi \rangle_A$ of the object is not destroyed; it is only transformed into coherence

$$| \Phi \rangle_{AB} = \sum_k F_B^k | \Phi \rangle_{AB}$$  \hspace{1cm} (47a)

with respect to the pointer observable $P_B \left( = \sum_k \alpha_k P_B^k \right)$ in the final premeasurement state $| \Phi \rangle_{AB}$ (cf (15) for the separate evolution of each branch). Though von Neumann did not expose a detailed theory of premeasurement, he did not consider the measurement paradox in the pure state case because the unitary evolution operator takes the pure composite state $| \phi \rangle_A | \phi \rangle_B$ into a pure state $| \Phi \rangle_{AB}$, and there is no way how coherence could disappear.

What von Neumann did was to assume that the initial state of the measuring instrument is in a mixed state $\rho_B^i$; a proper mixture due to incomplete knowledge about the state. Then one might conjecture that this mixture would lead to the mixture

$$\rho_{AB}^f = \sum_k \langle \phi \rangle_A^k E_A^k | \phi \rangle_A \times \left( F_B^f | \Phi \rangle_{AB} / ||F_B^f | \Phi \rangle_{AB}|| \right) \left( | \Phi \rangle_{AB}^i F_B^k / ||F_B^k | \Phi \rangle_{AB}|| \right)$$  \hspace{1cm} (47b)

(cf (14)), which is observed in the laboratory. Von Neumann disproves in detail this conjecture.

Von Neumann’s no-go theorem has been often considered as a proof of the measurement paradox, i.e., of the puzzle why unitary quantum mechanics does not furnish separate branches (terms in (47b)) for the individual measured objects. In particular, it is puzzling how a deterministic theory, as the one reviewed in this article, can lead to the indeterministic branches for the individual objects, which becomes deterministic (described by (47b)) on the ensemble level.

No other conceivable way (than the one treated in von Neumann’s no-go theorem) how one could obtain (47b) within unitary dynamics of measurement was seen till Everett 1957 and 1973 and De Witt 1973 shocked the world by hypothesizing that the separate branches of premeasurement might become parallel worlds and that we, and everything that we know, somehow become one of these worlds.

(III) The discrete observables $O_A = \sum_k \alpha_k E_A^k$ to which this review has been confined can be generalized to include also observables that contain a continuous part in their spectrum, in particular purely continuous observables as position and linear momentum.

**Literature** Busch and Lahti (1987), and also Busch, Grabowski, and Lahti (1995a) have investigated this subject.

(IV) The concept of observables that are measurable can be extended from ordinary ones to ones that are expressed as POV (positive-operator valued) measures (cf. Busch, Lahti, and Mittelstaedt, 1996). The PV (projector-valued) measures corresponding to ordinary observables are special cases.

**Literature** One should read section 3.6 in Part I of Vedral (2006). Generalized observables (POV measures) are described in detail in the book by Busch, Grabowski, and Lahti (1995b). Also the study in Busch, Kiukas, and Lahti (2008, section 3.), on connection between POV and PV measures based on the Neumark (1940) dilation theorem is recommended. Also the article Peres (1990) is relevant and interesting.

(V) The eigen-projectors $E_A^k$ of an ordinary observable can be generalized by positive operators, the physical meaning of which is ‘effects’. The projectors, with the meaning of events or properties or quantum statements, are special cases.

In the most general case, which is studied in Busch, Lahti, and Mittelstaedt 1996, premeasurement is defined by the probability reproducibility condition, and generally this is not equivalent to the calibration condition. The present investigation was undertaken in the hope that restriction to the physically most important case of ordinary discrete observables will help to delve deeper into the subject, obtain more results, and see them with more clarity and simplicity.

(VI) The theory presented in the present review is purely algebraic. The question of feasibility of the particular premeasurement procedures is not discussed.

**Literature** Concerning this important aspect of measurement, one should read Cassinelli and Lahti (1990) and section 6. in chapter III of Busch, Lahti, and Mittelstaedt 1996. One may also learn about the Wigner-Yanase-Araki theorem, which claims to set serious limitations on what can be measured exactly. One may read the critical short article Ohira and Pearle (1988), and the references therein.

(VII) Measurements cannot be performed without preparation, a procedure that brings about the initial states $| \phi \rangle_A$.

**Literature** One can read about preparation in section 8. of chapter III in Busch, Lahti, and Mittelstaedt 1996 or in Herbut 2001.

(VIII) Information-theoretical aspects of measurement theory have not been discussed in the present review either.

**Literature** Information gain in various kinds of measurement is investigated in Lahti, Busch, and Mittelstaedt 1991. One can read about this aspect in section 4. of chapter III of Busch, Lahti, and Mittelstaedt 1996. Entropic,
I am grateful to my onetime associates: the late Milan Vujčić, further Milan Damnjanović, Igor Ivanović, and Maja Burić for helpful and inspiring discussions on measurement theory.

Appendix Proof of an auxiliary algebraic certainty claim

We prove now the general claim that the following equivalence is valid for a pure state $|\psi\rangle$ and an event $E$:

$$\langle \psi | E | \psi \rangle = 1 \iff |\psi\rangle = E |\psi\rangle.$$  

Evidently

$$\langle \psi | E | \psi \rangle = 1 \Rightarrow \langle \psi | E^c | \psi \rangle = 0,$$

where $E^c \equiv I - E$ is the ortho-complementary projector and $I$ is the identity operator. Further, one has $||E^c |\psi\rangle|| = 0$, $E^c |\psi\rangle = 0$, and $E |\psi\rangle = |\psi\rangle$ as claimed. The inverse implication is obvious.

REFERENCES


Khrennikov, A., 2008, "The Role of Von Neumann and...


