The Quantum Wave Function as Property and as Pre-probability

Robert B. Griffiths *
Department of Physics, Carnegie-Mellon University,
Pittsburgh, PA 15213, USA

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Abstract

In quantum theory a wave function can correspond to a physical property in the sense of a one-dimensional subspace (ray) in the quantum Hilbert space. But in addition to this ontological role it can also be used in an epistemic sense as a pre-probability, a mathematical tool for assigning probabilities. Maintaining a clear distinction between these two uses is helpful in avoiding the trap of thinking of the collapse of the (epistemic) wave function as some sort of physical process.

1 Introduction

The following discussion is based on the consistent/decoherent histories, hereafter simply "histories", interpretation of quantum mechanics. For those unfamiliar with it there is a reasonably short introductions in [1], and a discussion of some of its conceptual difficulties in [2]. A detailed treatment is in [3]. Its proponents, including (obviously) me, regard it as "Copenhagen done right": textbook quantum mechanics with the confusing points properly explained, and the equivocations and arm waving deleted. The histories approach is based upon a consistent set of fundamental principles which do *not* include measurement, thus escaping Bell's complaints [4]. Consequently it has no measurement problem: measurements are simply physical processes subject to the same quantum principles that govern all physical processes, and analyzed using the same procedures. In addition, it is demonstrably *local*—the simple reason those mysterious nonlocal influences are unable to transport information over long distances (as everyone agrees) is that they do not exist. It resolves many paradoxes—in fact, I do not know of *any* of the standard quantum paradoxes that cannot be tamed, i.e., explained in rational terms, using the histories approach. Please let me know if you think there are exceptions, after first taking a look at the things analyzed in Chs. 19 to 25 of my book [3].

In a nutshell the distinctive features of the histories approach are as follows. Its ontology is based on subspaces of the Hilbert space, not hidden variables. Its dynamics is fundamentally random or stochastic, not just when measurements are made, but always. The deterministic Schrödinger equation is employed (without any additional stochastic terms added to it) to compute probabilities, not the future state of the world. And to use it one has to employ a mode of reasoning that differs in certain important respects from those we are accustomed to in classical physics. Quantum logic? No, not quite that bad, but also not that easy to swallow; what I have elsewhere called the New Quantum Logic [2].

2 Quantum Properties

In classical mechanics physical properties correspond to subsets of points in the classical phase space. In quantum mechanics as developed by von Neumann they correspond to *subspaces* of the Hilbert space.

*Electronic mail: rgrif@cmu.edu

(Sophisticates should add 'closed'; for my purposes the lowbrow approach of finite dimensional Hilbert spaces will suffice). The smallest subspace is just the zero ket, but that doesn't describe anything—it corresponds to the empty set in a classical phase space. The smallest interesting subspaces are one dimensional rays, and a single ray consists of all the multiples of some nonzero element $|\psi\rangle$. When expressed in a given representation $|\psi\rangle$ is a wave function. In what follows the term "wave function" will refer to a ket, without paying attention to its representation in real space or momentum space or spin space. A wavefunction when thought of in this way (i.e., as the corresponding ray) is the simplest example of a quantum property. There are other quantum properties associated with subspaces of dimension 2, 3, etc. It is convenient to identify any subspace with the corresponding projector (orthogonal projection operator). Given a normalized ket $|\psi\rangle$ the projector is $[\psi] = |\psi\rangle\langle\psi|$, where the square bracket provides a convenient shorthand. Of course $[\psi]$ determines $|\psi\rangle$ up to an arbitrary phase, which does not enter its physical interpretation, so "wave function" used loosely can also refer to $[\psi]$

Models based on standard probability theory, which is what we historians use, contain three things: a sample space $\mathcal{S}$ of mutually exclusive possibilities, a Boolean event algebra $\mathcal{E}$, and a probability measure $\mathcal{M}$. In the quantum case the sample space is a projective decomposition of the identity, a collection $\{P_j\}$ of mutually orthogonal projectors that sum to the identity:

$$I = \sum_j P_j; \quad P_j = P_j^\dagger = P_j^2; \quad P_j P_k = \delta_{jk} P_j. \tag{1}$$

The simplest example is an orthonoormal basis $\{|\phi^j\rangle\}$ with $P_j = [\phi^j]$.

For our purposes the Boolean event algebra $\mathcal{E}$ will consist of all projectors formed by taking sums of some of the projectors in the decomposition. If P be any element of $\mathcal{E}$ the property "not P' is given by the projector $I - P$, and if Q is another projector in $\mathcal{E}$, the combined property "P AND Q" is given by the product $PQ = QP$, and the disjunction "P OR Q" by $P + Q - PQ$. Everything is just fine and works in exactly the same way as in classical physics as long as all the properties correspond to projectors drawn from the same event algebra $\mathcal{E}$.

But in quantum mechanics there are many projectors which do not commute with other projectors. Thus for a spin-half particle the projectors corresponding to $S_z = \pm 1/2$ (in units of $\hbar$) are $[z^+]$ and $[z^-]$, which commute with each other (their product is zero) and which provide a decomposition of the identity $I = [z^+] + [z^-]$ for this two-dimensional Hilbert space. Similarly the projectors $[x^+]$ and $[x^-]$ corresponding to $S_x = \pm 1/2$ provide a different decomposition of the identity $I = [x^+] + [x^-]$. But none of the S_x projectors commute with the S_z projectors. What shall we do?

I know of three approaches to this problem. The first is to ignore it, which in my opinion is a prescription for another ninety years of quantum uncertainty. The second is that of quantum logic. As I understand it, this is a perfectly consistent way to deal with noncommuting projectors. But quantum logic as a mode of reasoning is so far removed from ordinary "classical" logic that no one has yet figured out how to use it to understand quantum mechanics and clean up the quantum paradoxes. At my university there are enthusiasts who think that superintelligent robots will be around in a decade or two (the time frame always seems to slip a bit). Perhaps these robots will be able to use quantum logic to understand the quantum world. Whether they will then be able to explain it to us (or even want to) is another question. But in the meantime there is a third approach, which I call the "new quantum logic" [2].

The central idea of the new quantum logic is to let PQ mean "P AND Q" in those cases, and *only* those cases, where $PQ = QP$, and thus the product (in either order) is a projector representing a quantum property. When the projectors do not commute, "P AND Q" is undefined, meaningless (the new quantum logic assigns it no meaning), it lies outside the language of discourse. (Think of it as analogous to the expression "P AND OR Q", which is meaningless in ordinary logic because it has not been put together according to the syntactical rules governing acceptable expressions.) Similarly, "P OR Q" is meaningless if the projectors P and Q do not commute. This idea generalizes to projective decompositions of the identity: two quantum sample spaces, $\{P_j\}$, $\{Q_k\}$ cannot be combined, must be kept strictly separate, unless they are *compatible*: all of the P projectors commute with all of the Q projectors. This is an instance of the *single framework rule.* (I use "framework" to refer to either a sample space or the associated event algebra; the context fixes the meaning.) I hasten to add that this rule does not deny to the theoretical physicist the option of constructing all sorts of incompatible sample spaces or frameworks when thinking about a problem; what is forbidden is *combining* incompatible sample spaces to make a single quantum description, or to frame a single logical argument. Ignoring the single framework rule is a good way to construct a paradox. The Bell-Kochen-Specker paradox is one example.

A quantum *history* is a sequence of quantum properties at a succession of times, say $t_1 < t_2 < \cdots t_f$; think of it as analogous to a random walk. Using a finite number of discrete times is adequate for the lowbrow physicist (me) and avoids technical complications. One can associate each history with a projector in a big Hilbert space of histories, and use a projective decomposition of the histories identity to set up a sample space, and from it an event algebra. See Sec. 4 of [1] or Sec. 8.3 of [3] for details.

3 Quantum Probabilities

As indicated above, in quantum mechanics a projective decomposition of the Hilbert space identity provides a sample space, both in the case of properties at a single time, and for stochastic time evolution, where the sample space consists of histories. Assigning probabilities, the $\mathcal{M}$ part of a probabilistic model, is done by choosing a nonnegative number $p(P_j)$ for each element of the sample space in such a way that these sum to 1. This determines the probability of any projector in the event algebra in the usual way: to $P_1 + P_2 + P_5$ assign the probability $p(P_1) + p(P_2) + p(P_5)$.

But how choose the $p(P_j)$ to begin with? The usual technique in classical probability theory is best thought of as guesswork, though often a very sophisticated sort of guesswork, especially if one is trying to fit experimental data. The same holds for quantum theory, with one extremely important exception. For the dynamics of a *closed* quantum system for which Schrödinger's equation provides a unitary time evolution, e.g.,

$$T(t_1, t_0) = \exp[-i(t_1 - t_0)H/\hbar] \tag{2}$$

if the Hamiltonian H is independent of time, the Born rule (and its extensions) assigns probabilities in a way which has (so far as I know) no analog in classical physics.

Consider how this works for a very special case which illustrates an epistemic use of a wave function as what I call a *pre-probability*. Let $\{|\phi^j\rangle\}$ be an orthonormal basis representing properties of interest at time t_0, and $\{|\hat{\phi}^k\rangle\}$ another (or possibly the same) basis at a time t_1. Question: if the system starts off in a particular state $|\phi^1\rangle$, which is to say it has the property $[\phi^1]$, at time t_0, what is the probability that it will have the property (be in the state) $|\hat{\phi}^k\rangle$ ($[\hat{\phi}^k]$)at a later time t_1?

The textbook approach is to solve Schrödinger's equation for $|\psi(t)\rangle$ as a function of t with the initial condition $|\psi(t_0)\rangle = |\phi^1\rangle$, thus

$$|\psi(t)\rangle = T(t, t_0)|\phi^1\rangle, \tag{3}$$

and then compute

$$\Pr(\hat{\phi}^k \mid \phi^1) = |\langle\hat{\phi}^k|\psi(t_1)\rangle|^2. \tag{4}$$

(The textbook is likely to say that this is the probability of the outcome of an appropriate *measurement* if one is carried out at the time in question, thereby confusing the poor student who wonders what measurements are doing in the middle of a physical theory of microscopic phenomena. In the histories approach the student can be told that there are competent experimentalists who know what they are doing and one of them can construct an apparatus to carry out a measurement in such a fashion that if the system with property $[\hat{\phi}^k]$ enters the apparatus at time t_1, after a suitable delay the pointer will be in position k, and thus the probability that the pointer is in position k at the end of the measurement is the same as the probability that the measured system had the corresponding property at the earlier time t_1. How the histories approach describes measurement process using proper quantum mechanical tools is discussed in Chs. 17 and 18 of [3].)

In (4) $|\psi(t_1)\rangle$ is what I call a *pre-probability*; i.e., a mathematical device used to calculate probabilities. This is an epistemic role, not an ontological role, as can be seen as follows. Our discussion has assumed that $|\phi^j\rangle$ and $|\bar{\phi}^k\rangle$, or to be more precise the corresponding projectors $[\phi^j]$ and $[\bar{\phi}^k]$, are ontological, (possible) quantum properties, and in particular we are interested in the probability that one of the $[\bar{\phi}^k]$ is actually the case at time t_1. This probability can equally well be obtained from the formula

$$\Pr(\hat{\phi}^k \mid \phi^1) = |\langle \phi^1 | \bar{\psi}^k(t_0) \rangle|^2. \tag{5}$$

where

$$|\bar{\psi}^k(t)\rangle = T(t, t_1)|\bar{\phi}^k\rangle, \tag{6}$$

i.e., one integrates Schrödinger's equation backwards in time. Observe that "the" wave function $|\psi(t)\rangle$ appears nowhere in (5) of (6). Yes, you will reply, but some other wave functions $|\bar{\psi}^k(t)\rangle$ are required. To which my answer is: true enough, but these are not at all the same as $|\psi(t)\rangle$, and I can calculate what I am interested in calculating, a probability distribution relating quantum properties at two different times, without making the slightest use of $|\psi(t)\rangle$. Thus $|\psi(t)\rangle$ is a mathematical tool which can be replaced by some other tool. A pre-probability is no more real, and perhaps even less real, than a probability, something which most of us would not put into the ontological category of "things".

4 Conclusion

Wave functions can play either an ontic or an epistemic role in quantum theory. In their ontic role they represent particular quantum properties, one-dimensional Hilbert subspaces, in the sense that $[\psi] = |\psi\rangle\langle\psi|$ is a projector onto a particular ray. (There are other quantum properties, subspaces of higher dimension, which do not correspond to any wavefunction; quantum ontology is not exhausted by wave functions.) In their epistemic role as pre-probabilities wave functions can be used to generate probabilities for this or that quantum property, and are best not thought of as more real than the probabilities they generate. (There are other quantum pre-probabilities, in particular, density operators.) Maintaining a distinction between wave functions as properties and as pre-probabilities is helpful in avoiding confusion. For example, wave function collapse is nothing to worry about; it is surely not a physical process, since a collapsing wave function is always a pre-probability. I myself prefer to avoid talk of wave function collapse, as one can always replace it by a discussion based on conditional probabilities.

Acknowledgments

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